

Characterizing the absolute continuity of the convolution of orbital measures in a classical Lie algebra

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ABSTRACT. Let \mathfrak{g} be a compact, simple Lie algebra of dimension d . It is a classical result that the convolution of any d non-trivial, G -invariant, orbital measures is absolutely continuous with respect to Lebesgue measure on \mathfrak{g} and the sum of any d non-trivial orbits has non-empty interior. The number d was later reduced to the rank of the Lie algebra (or rank +1 in the case of type A_n). More recently, the minimal integer $k = k(X)$ such that the k -fold convolution of the orbital measure supported on the orbit generated by X is an absolutely continuous measure was calculated for each $X \in \mathfrak{g}$.

In this paper \mathfrak{g} is any of the classical, compact, simple Lie algebras. We characterize the tuples (X_1, \dots, X_L) , with $X_i \in \mathfrak{g}$, which have the property that the convolution of the L -orbital measures supported on the orbits generated by the X_i is absolutely continuous and, equivalently, the sum of their orbits has non-empty interior. The characterization depends on the Lie type of \mathfrak{g} and the structure of the annihilating roots of the X_i . Such a characterization was previously known only for type A_n .

1. Introduction

Let G be a compact, connected simple Lie group and \mathfrak{g} its Lie algebra. Given $X \in \mathfrak{g}$, we let μ_X denote the G -invariant, orbital measure supported on O_X , the orbit generated by X under the adjoint action of G . Geometric properties of the Lie algebra ensure that if a suitable number of non-trivial orbits are added together the resulting subset of \mathfrak{g} has non-empty interior and if a suitable number of orbital measures are convolved together, the resulting measure is absolutely continuous with respect to the Lebesgue measure on \mathfrak{g} . From the work of Ragozin in [18] it can be seen that the dimension of the Lie algebra is a ‘suitable number’.

In a series of papers (see [9] and [10] and the papers cited therein) the authors, with various coauthors, improved upon Ragozin’s result determining, for each $X \in \mathfrak{g}$, the integer $k(X)$ with the property that μ_X^k is absolutely continuous for all $k \geq k(X)$ and μ_X^k is singular to Lebesgue measure otherwise (where μ_X^k denotes the

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k -fold convolution). Furthermore, the k -fold sum of O_X has non-empty interior if $k \geq k(X)$ and otherwise has measure zero. A formula was given for $k(X)$ depending on combinatorial properties of the annihilating roots of X . In particular, it was shown that the convolution of any r orbital measures is absolutely continuous if and only if r is at least the rank of the Lie algebras when \mathfrak{g} is of type B_n, C_n or D_n and r is at least $\text{rank}+1$ for the Lie algebras of type A_n . The proofs relied heavily upon representation theory and harmonic analysis.

By taking a geometric approach, Wright in [23] extended these results in the special case of the classical Lie algebra $\mathfrak{g} = su(n)$ (type A_{n-1}), proving that $\mu_{X_1} * \cdots * \mu_{X_L}$ is absolutely continuous with respect to Lebesgue measure if and only if $\sum_{i=1}^L s_i \geq n(L-1)$ where s_i is the dimension of the largest eigenspace of the $n \times n$ matrix X_i , provided it is not the case that $L = 2$, $n \geq 4$ is even, and X_1, X_2 each have two distinct eigenvalues, both of multiplicity $n/2$.

Using primarily algebraic methods, Graczyk and Sawyer (c.f., [4], [5]), addressed analogous problems in the setting of a non-compact, symmetric space, improving upon other work of Ragozin, [17]. In particular, they characterized when the convolution of two (possibly different) bi-invariant measures is absolutely continuous in the symmetric spaces $sl(n, F)/su(n, F)$ (where the restricted root system is also type A_{n-1}).

Inspired by their methods, in this paper we characterize the L -tuples, (X_1, \dots, X_L) with $X_i \in \mathfrak{g}$, such that the convolution $\mu_{X_1} * \cdots * \mu_{X_L}$ is absolutely continuous when the Lie algebra is any one of the classical Lie algebras (those of type A_n, B_n, C_n or D_n), leaving only one pair in D_n where we have been unable to decide the answer. As well, this characterizes the L -tuples such that $\sum_{i=1}^L O_{X_i}$ has non-empty interior in \mathfrak{g} as opposed to measure zero. As Wright found with type A_n , the characterization can be expressed most simply as a function of the dimensions of the largest eigenspaces of the X_i when these are viewed as matrices in the classical matrix Lie algebras (see Section 3 for the precise statement). The characterization can also be described in terms of the root structure of the set of annihilating roots of the X_i , as was done in the previous study of convolutions of a single orbital measure. Our argument is completely different from that used by Wright and from the harmonic analysis - representation theory approach used by the authors previously. It relies heavily upon the (algebraic) Lie theory of roots and root vectors.

Using these results, we also obtain a similar characterization of the absolute continuity of the convolution products of G -invariant measures, μ_{x_i} , supported on conjugacy classes C_{x_i} in G , for the elements $x_i \in G$ whose annihilating roots agree with those of a preimage of x_i in \mathfrak{g} under the exponential map. This extends work of [8] where the minimal integer $k(x)$ with the property that $\mu_x^{k(x)}$ is absolutely continuous was determined.

In a future paper, we will adapt our general strategy to improve upon Graczyk and Sawyer's symmetric space results.

Finding the density function, or Radon Nikodym derivative, of the absolutely continuous measure $\mu_{X_1} * \cdots * \mu_{X_L}$ is a challenging problem. In the case of the convolution of two orbital measures in $su(n)$, this has been computed in [2]. A general formula for the convolution of two orbital measures in terms of the projection of such measures to maximal tori was found in [1]. The density function for the analogous problem on non-compact symmetric spaces was studied in [3] (and see also the references cited there). In [15], the sum of two adjoint orbits in $su(n)$

is explicitly described in terms of a system of linear equations, but for more than 2-fold sums this too seems very difficult. Other work investigating the smoothness properties of convolutions of measures supported on manifolds whose product has non-empty interior was carried out by Ricci and Stein in [19] and [20].

The paper is organized as follows: In section 2 we review background material in Lie theory and introduce basic notation. In section 3 we state the main result. The necessity of our characterization is proven in section 4. In section 5 we establish the general strategy for tackling the absolute continuity problem and then complete the proof of the main theorem in section 6. In section 7 we discuss consequences of our result and deduce the absolute continuity result for convolutions of orbital measures on Lie groups mentioned above.

2. Notation and Background

2.1. Notation. We begin by establishing notation and reviewing basic facts about roots and root vectors. Assume G_n is a classical, compact, connected simple Lie group of rank n , one of type A_n , B_n , C_n or D_n . We denote by \mathfrak{g}_n its (real) Lie algebra, \mathfrak{t}_n a maximal torus of \mathfrak{g}_n and W the Weyl group.

We write $[\cdot, \cdot]$ for the Lie bracket action. The map $ad : \mathfrak{g}_n \rightarrow \mathfrak{g}_n$ is given by $ad(X)(Y) = [X, Y]$. The exponential function, \exp , is a surjection of \mathfrak{g}_n onto G_n , and G_n acts on \mathfrak{g}_n by the adjoint action, denoted $Ad(\cdot)$. Recall that for $M \in \mathfrak{g}_n$,

$$Ad(\exp M) = \exp(ad(M)) = Id + \sum_{k=1}^{\infty} \frac{ad^k(M)}{k!}$$

where $ad^k(M)$ is the k -fold composition of $ad(M)$.

By an orbit of an element $X \in \mathfrak{g}_n$, we mean the subset

$$O_X := \{Ad(g)(X) : g \in G_n\} \subseteq \mathfrak{g}_n.$$

There is no loss in assuming X belongs to \mathfrak{t}_n since every orbit contains a torus element. Orbits are compact manifolds of proper dimension in \mathfrak{g}_n and hence of Lebesgue measure zero. If $X = 0$, then $O_X = \{0\}$ is a singleton, but otherwise O_X has positive dimension.

By the orbital measure, μ_X , we mean the probability measure invariant under the adjoint action of G_n and compactly supported on O_X . It integrates bounded, continuous functions f on \mathfrak{g}_n by the rule

$$\int_{\mathfrak{g}_n} f d\mu_X = \int_{G_n} f(Ad(g)X) dg$$

where dg is the Haar measure on G_n . The orbital measures are singular to Lebesgue measure since their supports have Lebesgue measure zero. Except in the special case when $X = 0$, μ_X is an example of a continuous measure, meaning the μ_X - measure of any singleton is zero.

The classical Lie groups and algebras are said to be of type A_n for $n \geq 1$, B_n for $n \geq 2$, C_n for $n \geq 3$ or D_n for $n \geq 4$. This means that the root system of the complexified Lie algebra with respect to the complexified torus, denoted Φ_n , is of that Lie type. It is often convenient to refer to type A_n as type $SU(n+1)$ for reasons that will become clear later.

For the convenience of the reader we describe Φ_n below for each of the classical types. Note that by e_j we mean the j 'th standard basis vector of \mathbb{R}^n (or in \mathbb{R}^{n+1} in the case of type A_n). The real span of Φ_n , denoted $sp\Phi_n$, is equal to \mathbb{R}^n (or the

subspace of \mathbb{R}^{n+1} spanned by the standard vectors $e_j - e_{n+1}$ for $j = 1, \dots, n$ in the case of type A_n).

Lie algebra	Root system Φ_n
A_n	$\{\pm(e_i - e_j) : 1 \leq i < j \leq n+1\}$
B_n	$\{\pm e_i, \pm e_i \pm e_j : 1 \leq i \neq j \leq n\}$
C_n	$\{\pm 2e_i, \pm e_i \pm e_j : 1 \leq i \neq j \leq n\}$
D_n	$\{\pm e_i \pm e_j : 1 \leq i \neq j \leq n\}$

In the case of type A_n , the Weyl group is the group of permutations on the letters $\{1, \dots, n+1\}$. For types B_n , C_n (and D_n), the Weyl groups are the group of permutations on $\{1, \dots, n\}$, together with (an even number of) sign changes.

These Lie algebras and groups can be identified with the classical matrix algebras and groups listed below. All compact, connected simple Lie groups are homomorphic images by finite subgroups of these classical matrix groups.

- $su(n)$ – the set of $n \times n$ skew-Hermitian, trace zero matrices is the model we use for the Lie algebra of type A_{n-1} . $SU(n)$ – the $n \times n$ special unitary matrices is a compact Lie group of type A_{n-1} .
- $so(p)$ – the set of $p \times p$ real, skew-symmetric matrices. When $p = 2n$ it is the Lie algebra of type D_n and when $p = 2n+1$ it is of type B_n . $SO(p)$ – the $p \times p$ special orthogonal matrices are associated compact Lie groups.
- $sp(n)$ – the set of $2n \times 2n$ matrices of the form $\begin{bmatrix} A & B \\ -\overline{B} & A \end{bmatrix}$ where A, B are complex $n \times n$ matrices with B symmetric and A skew-Hermitian is the Lie algebra of type C_n . The n 'th order symplectic group, $Sp(n)$, is the set of $2n \times 2n$ unitary matrices U satisfying $U^{tr}JU = J$, where $J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$ with I being the $n \times n$ identity matrix. $Sp(n)$ is a compact Lie group of type C_n .

For each root $\alpha \in \Phi_n$, we let E_α denote a corresponding root vector so that if $H \in \mathfrak{t}_n$, then

$$(2.1) \quad [H, E_\alpha] = i\alpha(H)E_\alpha.$$

(We make the convention that roots are real valued.) We will choose a collection of root vectors, $\{E_\alpha\}$, that form a Weyl basis (see [21, p. 290]). In particular, this ensures that if α, β and $\alpha + \beta$ are roots, then there are non-zero scalars $N_{\alpha, \beta}$ satisfying $N_{\alpha, \beta} = N_{-\alpha, -\beta}$ and

$$[E_\alpha, E_\beta] = N_{\alpha, \beta}E_{\alpha+\beta}.$$

If $\alpha + \beta$ is not a root, then $[E_\alpha, E_\beta] = 0$.

The root vector, E_α , can be written in a unique way as $E_\alpha = RE_\alpha + iIE_\alpha$, where RE_α and IE_α both belong to the (real) Lie algebra \mathfrak{g}_n . We refer to these as the real and imaginary parts of the root vector. We write FE_α if we mean either RE_α or IE_α . One can easily see that $E_{-\alpha} = RE_\alpha - iIE_\alpha$. Furthermore, $RE_\alpha = (E_\alpha + E_{-\alpha})/2$ and $IE_\alpha = (E_\alpha - E_{-\alpha})/(2i)$.

The vector space spanned by RE_α and IE_α over various sets of roots α will be important to us. In particular, we put

$$(2.2) \quad \mathcal{V}_n = \{RE_\alpha, IE_\alpha : \alpha \in \Phi_n^+\} \subseteq \mathfrak{g}_n$$

where Φ_n^+ denotes the subset of positive roots. With this notation the Lie algebra can be decomposed as

$$\mathfrak{g}_n = \mathfrak{t}_n \bigoplus_{\alpha \in \Phi_n^+} sp\{RE_\alpha, IE_\alpha\} = \mathfrak{t}_n \bigoplus sp\mathcal{V}_n$$

where sp denotes the real span. Thus the dimension of \mathfrak{g}_n is equal to $n + |\Phi_n|$.

From (2.1) it follows that

$$(2.3) \quad [H, RE_\alpha] = -\alpha(H)IE_\alpha \text{ and } [H, IE_\alpha] = \alpha(H)RE_\alpha.$$

It is also well known that

$$[RE_\alpha, IE_\alpha] = \frac{-1}{2i}[E_\alpha, E_{-\alpha}]$$

is a non-zero element of the maximal torus. It should be noted that if $\{\alpha_j : j \in J\} \subseteq \Phi_n$ is a spanning set for $sp\Phi_n$, then $\{RE_{\alpha_j}, IE_{\alpha_j} : j \in J\}$ spans \mathfrak{t}_n .

Since $\{E_\alpha\}$ is a Weyl basis, we have

$$(2.4) \quad \begin{aligned} [RE_\alpha, RE_\beta] &= cRE_{\alpha+\beta} + dRE_{\beta-\alpha}, \\ [RE_\alpha, IE_\beta] &= cIE_{\alpha+\beta} + dIE_{\beta-\alpha} \\ [IE_\alpha, IE_\beta] &= -cRE_{\alpha+\beta} + dRE_{\beta-\alpha}, \end{aligned}$$

where RE_γ and IE_γ should be understood to be the zero vector if γ is not a root and $c = N_{\alpha,\beta}/2$, $d = N_{\alpha,-\beta}/2$.

We refer the reader to [12], [14] and [21] for proofs of these well known facts and further details on the representation theory of Lie algebras.

2.2. Annihilating roots. We call a root, α , an *annihilating root* of $X \in \mathfrak{t}_n$ if $\alpha(X) = 0$ and call α a *non-annihilating root* of X otherwise. The *set of annihilating roots* of X ,

$$\Phi_X := \{\alpha \in \Phi : \alpha(X) = 0\},$$

is a root subsystem of Φ_n . As we will see, these root subsystems are critical for understanding properties about orbits and orbital measures, as are the associated root vectors. We will denote by

$$(2.5) \quad \mathcal{N}_X := \{RE_\alpha, IE_\alpha : \alpha \notin \Phi_X\} \subseteq \mathcal{V}_n,$$

the linearly independent subset of \mathcal{V}_n consisting of the real and imaginary parts of the root vectors corresponding to the non-annihilating roots of X . It is known that $\dim O_X = |\mathcal{N}_X|$ [16, VI.4]. Indeed, the tangent space at X to O_X is spanned by the vectors in \mathcal{N}_X and these are linearly independent (see the proof of Prop. 1).

2.3. Type of an Element. The torus of $su(n)$, the classical Lie algebra of type A_{n-1} (or type $SU(n)$) consists of the diagonal matrices in $su(n)$. After applying a suitable Weyl conjugate, any X in the torus can be identified with the n -vector of the real parts of the diagonal elements,

$$X = (\underbrace{a_1, \dots, a_1}_{s_1}, \dots, \underbrace{a_m, \dots, a_m}_{s_m}),$$

where the $a_j \in \mathbb{R}$ are distinct and $\sum_{j=1}^m s_j a_j = 0$. This means that ia_j is an eigenvalue of the $n \times n$ matrix X with multiplicity s_j . The set of annihilating roots

of X is $\Phi_X = \Psi_1 \cup \dots \cup \Psi_m$ where

$$\Psi_1 = \{e_i - e_j : 1 \leq i \neq j \leq s_1\} \text{ and}$$

$$\Psi_l = \{e_i - e_j : s_1 + \dots + s_{l-1} < i \neq j \leq s_1 + \dots + s_l\} \text{ for } l > 1.$$

Following [9], we say that X is type $SU(s_1) \times \dots \times SU(s_m)$ as this is the Lie type of its set of annihilating roots.

The torus of $so(2n+1)$, the classical Lie algebra of type B_n , consists of block diagonal matrices, with n 2×2 blocks of the form $\begin{bmatrix} 0 & b_j \\ -b_j & 0 \end{bmatrix}$ having $b_j \geq 0$, and a 0 in the final diagonal position. We identify X in the torus with the n -vector $(b_1, \dots, b_n) \in \mathbb{R}^{+n}$. Up to a Weyl conjugate, X can thus be identified with the n -vector

$$(2.6) \quad X = (\underbrace{0, \dots, 0}_J, \underbrace{a_1, \dots, a_1}_{s_1}, \dots, \underbrace{a_m, \dots, a_m}_{s_m})$$

where the $a_j > 0$ are distinct. One can see that 0 is an eigenvalue of the $(2n+1) \times (2n+1)$ matrix X with multiplicity $2J+1$ and $\pm ia_j$ are eigenvalues with multiplicity s_j .

The set of annihilating roots $\Phi_X = \Psi_0 \cup \Psi_1 \dots \cup \Psi_m$ where

$$\Psi_0 = \{\pm e_k, \pm e_i \pm e_j : 1 \leq i, j, k \leq J, i \neq j\} \text{ and}$$

$$\Psi_l = \{e_i - e_j : J + s_1 + \dots + s_{l-1} < i \neq j \leq J + s_1 + \dots + s_l\}$$

for $l = 1, \dots, m$. We will say that X is *type*

$$B_J \times SU(s_1) \times \dots \times SU(s_m),$$

as this is the Lie type of Φ_X . Here by B_1 we mean the root subsystem $\{\pm e_1\}$, while $SU(1)$, B_0 and $SU(0)$ are empty (and typically omitted in the description).

Similarly, if X belongs to the torus of the Lie algebra of type C_n or D_n then, up to a Weyl conjugate, X can be identified with the n -vector

$$X = (\underbrace{0, \dots, 0}_J, \underbrace{a_1, \dots, a_1}_{s_1}, \dots, \underbrace{a_m, \dots, (\pm)a_m}_{s_m})$$

where the $a_j > 0$ are distinct. We remark that the minus sign is needed only in type D_n and only if $J = 0$. (This is because the Weyl group in type D_n changes only an even number of signs.) Viewing X as an $2n \times 2n$ matrix in $sp(n)$ or $so(2n)$, this means that 0 is an eigenvalue of X with multiplicity $2J$, and $\pm ia_j$ are eigenvalues with multiplicity s_j .

The set of annihilating roots of X can again be written as $\Phi_X = \Psi_0 \cup \Psi_1 \dots \cup \Psi_m$. In this case

$$\Psi_0 = \{\pm 2e_k, \pm e_i \pm e_j : 1 \leq i, j, k \leq J, i \neq j\}$$

when the Lie algebra is type C_n and

$$\Psi_0 = \{\pm e_i \pm e_j : 1 \leq i, j \leq J, i \neq j\}$$

when the Lie algebra is type D_n . For $l \geq 1$, the Ψ_l are as in type B_n , except when $X = (a_1, \dots, a_1, \dots, a_m, \dots, -a_m)$ in D_n when

$$\Psi_m = \{\pm(e_i - e_j), \pm(e_i + e_n) : n - s_m < i \neq j \leq n - 1\}.$$

We will say X is *type*

$$C_J \times SU(s_1) \times \dots \times SU(s_m) \text{ or } D_J \times SU(s_1) \times \dots \times SU(s_m)$$

respectively, as these are the Lie types of Φ_X . Here C_1 is the subsystem $\{\pm 2e_1\}$, C_2 is $\{\pm 2e_1, \pm 2e_2, \pm e_1 \pm e_2\}$, D_2 is $\{\pm e_1 \pm e_2\}$ (or type $A_1 \times A_1$), D_3 is defined in the obvious way, and D_1, D_0, C_0 are empty (and often omitted).

Note that there are two distinct subsystems (up to Weyl conjugacy) of annihilating roots of elements of type $SU(n)$ in D_n .

DEFINITION 1. Suppose X is in the torus of the Lie algebra of type B_n and is type $B_J \times SU(s_1) \times \cdots \times SU(s_m)$. We will say X is **dominant B type** if $2J \geq \max s_j$, and is **dominant SU type** otherwise. We define **dominant C and D type** similarly for X in C_n or D_n .

It was shown in [9, Thm. 8.2] that for each non-zero $X \in \mathfrak{g}_n$, there is an integer $k(X)$ such that for $k \geq k(X)$, $\mu_X^k \in L^1 \cap L^2(\mathfrak{g}_n)$ (in particular, μ_X^k is absolutely continuous with respect to Lebesgue measure) and μ_X^k is purely singular if $k < k(X)$. A formula was given for $k(X)$ depending only on the type of X and the type of the Lie algebra. For example, if X is dominant SU type in the Lie algebra of type B_n, C_n or D_n , and not of type $SU(n)$ when the Lie algebra is type D_n , then $k(X) = 2$. If X is type $B_{n-1}, (C_{n-1}, D_{n-1} \text{ or } SU(n-1))$ in the Lie algebra of type $B_n (C_n, D_n \text{ or } SU(n))$, then $k(X) = n$ and this is the maximal choice required for $k(X)$.

3. Statement of the Main Result

3.1. Eligible and Exceptional Tuples. We introduce the following terminology.

NOTATION 1. If X is of type $SU(s_1) \times \cdots \times SU(s_m)$ in the Lie algebra of type A_n , put $S_X = \max s_j$.

If X is type $B_J \times SU(s_1) \times \cdots \times SU(s_m)$ in the Lie algebra of type B_n , put

$$S_X = \begin{cases} 2J & \text{if } X \text{ is dominant } B \text{ type} \\ \max s_j & \text{else} \end{cases}$$

Define S_X similarly when X belongs to the Lie algebras of type C_n or D_n .

If $X \in \mathfrak{so}(2n+1)$ is dominant B type, then the dimension of the largest eigenspace of the matrix X is $S_X + 1$, while if X is dominant SU type, then the dimension of the largest eigenspace is S_X . In all the other Lie algebras, S_X is the dimension of the largest eigenspace when X is viewed as a matrix in the appropriate classical matrix algebra.

DEFINITION 2. (i) We will say that the L -tuple (X_1, X_2, \dots, X_L) of elements in the torus of a Lie algebra of type $SU(n+1)$ is **eligible** in \mathfrak{g}_n if

$$\sum_{i=1}^L S_{X_i} \leq (L-1)(n+1).$$

(ii) We will say that the L -tuple (X_1, X_2, \dots, X_L) of elements in the torus of a Lie algebra of type B_n, C_n or D_n is **eligible** in \mathfrak{g}_n if

$$(3.1) \quad \sum_{i=1}^L S_{X_i} \leq (L-1)2n.$$

DEFINITION 3. We will say that $(X_1, X_2, \dots, X_L) \in \mathfrak{t}^L$ is an **exceptional tuple** if it is any one of the following:

- \mathfrak{g} is type $SU(2n)$, $L = 2$, $n \geq 2$ and X_1 and X_2 are both of type $SU(n) \times SU(n)$ (i.e., $X_i = (\underbrace{a_i, \dots, a_i}_n, \underbrace{-a_i, \dots, -a_i}_n)$);
- \mathfrak{g} is type D_n , $L = 2$, X_1 is type $SU(n)$ and X_2 is either type $SU(n)$ or type $SU(n-1)$ (more precisely, type $SU(n-1) \times D_1$ or $SU(n-1) \times SU(1)$);
- \mathfrak{g} is type D_4 , $L = 2$, X_1 is type $SU(4)$ and X_2 is either type $SU(2) \times SU(2)$ and Φ_{X_2} is Weyl conjugate to a subset of Φ_{X_1} , or X_2 is type $SU(2) \times D_2$;
- \mathfrak{g} is type D_4 , $L = 3$ and X_1, X_2, X_3 are all of type $SU(4)$ with Weyl conjugate sets of annihilators.

DEFINITION 4. We will call (X_1, X_2, \dots, X_L) an **absolutely continuous tuple** if $\mu_{X_1} * \mu_{X_2} * \dots * \mu_{X_L}$ is an absolutely continuous measure.

Our main result is that other than for the exceptional tuples, eligibility characterizes absolute continuity of the convolution product. The proof of this theorem will occupy most of the remainder of the paper. Here is the formal statement of the theorem.

3.2. Main Result.

THEOREM 1. Let \mathfrak{g}_n be one of the classical, compact, connected Lie algebras of type A_n with $n \geq 1$, B_n with $n \geq 2$, C_n with $n \geq 3$, or D_n with $n \geq 4$. Assume non-zero X_i , $i = 1, 2, \dots, L$ for $L \geq 2$, belong to the torus of \mathfrak{g}_n .

(i) Suppose (X_1, X_2, \dots, X_L) is not an exceptional tuple. The measure, $\mu_{X_1} * \mu_{X_2} * \dots * \mu_{X_L}$, is absolutely continuous with respect to Lebesgue measure on \mathfrak{g}_n if and only if (X_1, X_2, \dots, X_L) is an eligible tuple.

(ii) If (X_1, X_2, \dots, X_L) is an exceptional tuple, other than a pair (X_1, X_2) of type $(SU(n), SU(n-1))$ ¹ in a Lie algebra of type D_n with $n \geq 6$, then the measure $\mu_{X_1} * \mu_{X_2} * \dots * \mu_{X_L}$ is not absolutely continuous.

REMARK 1. The characterization of absolute continuity in type A_n was previously established by Wright [23]. We will include a proof in this paper as our approach is completely different and requires little additional effort.

REMARK 2. (i) We conjecture that a pair of type $(SU(n), SU(n-1))$ in D_n with $n \geq 6$ also fails to be absolutely continuous.

(ii) Notice that unlike the case for convolutions of the same orbital measure ([9, Thm. 8.2]), the property of being absolutely continuous does not depend only upon the type of the annihilating root systems of the underlying elements, but also, in some cases, upon their Weyl conjugacy class.

In proving both absolute continuity and its failure we will rely crucially upon the following known geometric properties.

The notation $T_Z(O_X)$ will denote the tangent space to O_X at $Z \in O_X$.

PROPOSITION 1. The measure $\mu_{X_1} * \mu_{X_2} * \dots * \mu_{X_L}$ on \mathfrak{g}_n is absolutely continuous with respect to Lebesgue measure if and only if any of the following hold:

- (i) $\sum_{i=1}^L O_{X_i} \subseteq \mathfrak{g}_n$ has non-empty interior;

¹When we say a pair (X, Y) is of type $(*, **)$ we mean that X is of type $*$ and Y is of type $**$.

- (ii) $\sum_{i=1}^L O_{X_i} \subseteq \mathfrak{g}_n$ has positive Lebesgue measure;
 (iii) There exists $g_i \in G_n$ with $g_1 = Id$, such that
 (3.2) $sp\{Ad(g_i)(\mathcal{N}_{X_i}) : i = 1, \dots, L\} = \mathfrak{g}_n$,
 (iv) There exists $g_i \in G_n$ with $g_1 = Id$, such that

$$\sum_{i=1}^L T_{Ad(g_i)(X_i)}(O_{X_i}) = \mathfrak{g}_n.$$

Furthermore, if the identity holds in (iii) or (iv) for one choice of $(g_2, \dots, g_L) \in G_n^{L-1}$, then it holds for all (g_2, \dots, g_L) in an open dense subset of G_n^{L-1} of full measure.

REMARK 3. We note that (ii) implies that if $\mu_{X_1} * \mu_{X_2} * \dots * \mu_{X_L}$ is not absolutely continuous, then $\mu_{X_1} * \mu_{X_2} * \dots * \mu_{X_L}$ is a purely singular measure.

PROOF. This proposition is a compilation of arguments that can be found in [6], [9] and [18]. We include a sketch here for the convenience of the reader. We will show that (iii) and (iv) are equivalent and then demonstrate the implications (ii) \Rightarrow (iv) \Rightarrow absolute continuity and (iv) \Rightarrow (i). If $\mu_{X_1} * \mu_{X_2} * \dots * \mu_{X_L}$ is absolutely continuous or (i) holds, then (ii) clearly holds so this completes the equivalence.

(iii) \Leftrightarrow (iv). It is well known (see [6], [16, VI.4]) that

$$T_X(O_X) = \{[Y, X] : Y \in \mathfrak{g}_n\}.$$

Writing $Y = \sum a_\alpha RE_\alpha + b_\alpha IE_\alpha + t$ for some $t \in \mathfrak{t}_n$ and a_α, b_α real, it is easily seen that $T_X(O_X) = sp\mathcal{N}_X$. Further, $T_{Ad(g)X}(O_x) = Ad(g)(T_X(O_X)) = sp\{Ad(g)\mathcal{N}_x\}$, proving the equivalence of (iii) and (iv).

The final comment is an analyticity argument. Assume (iii) holds, for example, with $g = (Id, g_2, \dots, g_L)$. For any $h = (h_1, h_2, \dots, h_L) \in G_n^L$, $h_1 = Id$, consider the collection $Ad(h_j)Y$ for $Y \in \mathcal{N}_{X_j}$ and $j = 1, \dots, L$, as vectors in $\mathbb{R}^{\dim \mathfrak{g}_n}$, and form the associated matrix $M(h)$. As (iii) holds with g , there is a suitable square submatrix of $M(g)$ with non-zero determinant. By analyticity of the determinant map, the determinant of the corresponding square submatrix of $M(h)$ must be non-zero for an open, dense subset of $h \in G_n^{L-1}$ of full measure. The same argument applies to (iv).

(ii) \Rightarrow (iv). Consider the addition map $F : O_{X_1} \times \dots \times O_{X_L} \rightarrow \mathfrak{g}_n$ given by $F(Y_1, \dots, Y_L) = \sum_{j=1}^L Y_j$. The image of F is $\sum_{j=1}^L O_{X_j}$. If the rank of F is not full at any point in its domain, then Sard's theorem ([13, p. 286]) implies the measure of the image of F is zero. Thus the differential of F at some point $Y = (Y_1, \dots, Y_L)$, where $Y_j = Ad(g_j)X_j$, has full rank. But the range of the differential of F at Y is $\sum_{j=1}^L T_{Y_j}(O_{X_j})$ and hence this sum must be \mathfrak{g}_n .

(iv) \Rightarrow (i). The hypothesis of (iv) guarantees that the map F defined above has full rank at some point Y . By the Implicit function theorem, F is an open map in a neighbourhood of Y and thus $\text{Im } F$ has non-empty interior.

(iv) \Rightarrow absolute continuity. This is similar again. To see that the measure $\mu = \mu_{X_1} * \mu_{X_2} * \dots * \mu_{X_L}$ is absolutely continuous with respect to m , we should show that $\mu(E) = 0$ whenever $m(E) = 0$. Define $f : G_n^L \rightarrow \mathfrak{g}_n$ by

$$f(g_1, \dots, g_L) = F(Ad(g_1)X_1, \dots, Ad(g_L)X_L).$$

By definition, $\mu(E) = m_{G_n^L}(f^{-1}(E))$. By (iv), the differential of f has full rank at some point. An analyticity argument ensures that this is true on a subset of $g \in G_n^L$

of full measure. An application of the Implicit function theorem shows $f^{-1}(E)$ has $m_{G_n^L}$ -measure zero. For more details see [18, Thm. 2.2]. \square

An immediate corollary of this proposition and the main theorem is the following.

COROLLARY 1. *Suppose (X_1, X_2, \dots, X_L) is eligible and not exceptional. Then $\sum_{i=1}^L O_{X_i}$ has non-empty interior. If (X_1, X_2, \dots, X_L) is either not eligible or is exceptional and not type $(SU(n), SU(n-1))$ in D_n , then $\sum_{i=1}^L O_{X_i}$ has measure zero.*

There is a sufficient condition for absolute continuity, established by Wright in [23], that we will use in the proof of the main theorem to establish the absolute continuity of certain convolution products of orbital measures in small rank Lie algebras. We state this result below. By the rank of a subsystem we mean the dimension of the vector space it spans.

THEOREM 2. [23, Thm. 1.3] *Let X_1, \dots, X_L belong to the torus of \mathfrak{g}_n . Assume*

$$(3.3) \quad (L-1)(|\Phi| - |\Psi|) - 1 \geq \sum_{i=1}^L \left(|\Phi_{X_i}| - \min_{\sigma \in W} |\Phi_{X_i} \cap \sigma(\Psi)| \right)$$

*for all root subsystems $\Psi \subseteq \Phi$ of rank $n-1$ and having the property that $sp(\Psi) \cap \Phi = \Psi$. Then $\mu_{X_1} * \dots * \mu_{X_L}$ is absolutely continuous.*

4. Tuples That Are Not Absolutely Continuous

We begin by establishing the necessity of the conditions which give absolute continuity.

4.1. Eligibility is a requirement for absolute continuity.

LEMMA 1. *If (X_1, \dots, X_L) is an absolutely continuous L -tuple, then (X_1, \dots, X_L) is eligible.*

PROOF. Suppose the L -tuple, $(X_1, \dots, X_L) \in \mathfrak{g}_n^L$, is not eligible, that is,

$$\sum_{i=1}^L S_{X_i} \geq (L-1)2n+1 \text{ (or } (L-1)(n+1)+1 \text{ if } \mathfrak{g}_n \text{ is type } A_n.)$$

Let α_i be the eigenvalue of X_i with greatest multiplicity (where we view each X_i as a complex matrix of the appropriate size depending on the Lie type of \mathfrak{g}_n) and let g_i belong to the associated Lie group, G_n . Let V_i be the eigenspace of $Ad(g_i)(X_i)$ corresponding to the eigenvalue α_i .

If \mathfrak{g}_n is of type C_n or D_n , then $Ad(g_i)(X_i)$ are $2n \times 2n$ matrices and $\dim V_i = S_{X_i}$, so

$$\sum_{i=1}^L \dim V_i = \sum_{i=1}^L S_{X_i} \geq (L-1)2n+1.$$

We deduce that

$$\begin{aligned}
& \dim \bigcap_{i=1}^L V_i \\
&= \sum_{i=1}^L \dim V_i - \left(\dim(V_1 + V_2) + \dim((V_1 \cap V_2) + V_3) + \cdots + \dim\left(\bigcap_{i=1}^{L-1} V_i + V_L\right) \right) \\
&\geq (L-1)2n + 1 - 2n(L-1) \geq 1,
\end{aligned}$$

and hence the matrices, $Ad(g_i)(X_i)$, have a common eigenvector, v . As

$$\sum_{i=1}^L Ad(g_i)(X_i)(v) = \sum_{i=1}^L \alpha_i v,$$

it follows that $\sum_i \alpha_i$ is an eigenvalue of $\sum_i Ad(g_i)(X_i)$. Since $\sum_i Ad(g_i)(X_i)$ is an arbitrary element of $O_{X_1} + \cdots + O_{X_L}$, one can see that every element of $\sum_i O_{X_i}$ has eigenvalue $\sum_i \alpha_i$. This is impossible if $O_{X_1} + \cdots + O_{X_L}$ has non-empty interior, thus an application of Prop. 1(i) allows us to conclude that $\mu_{X_1} * \cdots * \mu_{X_L}$ is not absolutely continuous.

The argument is similar if \mathfrak{g}_n is type A_n , viewing X_i as matrices in $su(n+1)$, acting on \mathbb{R}^{n+1} .

In the case when \mathfrak{g}_n is type B_n we require a slight variation on the argument since every matrix in the Lie algebra $so(2n+1)$ (the model for type B_n) has 0 as an eigenvalue. We use the same notation as above and first observe that if all X_i are dominant B type, then all $\alpha_i = 0$ and $\dim V_i = S_{X_i} + 1$. Thus $\sum_{i=1}^L \dim V_i \geq (L-1)2n + L + 1$. Since the vector spaces V_i are subspaces of \mathbb{R}^{2n+1} , it follows that

$$\dim \bigcap_{i=1}^L V_i \geq (L-1)2n + L + 1 - (2n+1)(L-1) \geq 2.$$

Consequently, 0 is an eigenvalue of every element of $O_{X_1} + \cdots + O_{X_L}$ of multiplicity at least two. Again, we can conclude that $O_{X_1} + \cdots + O_{X_L}$ has empty interior and therefore (X_1, \dots, X_L) is not an absolutely continuous tuple.

If, instead, precisely one X_i is dominant SU type, with eigenvalue $\alpha \neq 0$ of maximum multiplicity, then $\sum \dim V_i \geq (L-1)2n + L$. This shows that $\dim \bigcap_{i=1}^L V_i$ has dimension at least one and hence every element of $O_{X_1} + \cdots + O_{X_L}$ has α as an eigenvalue, again a contradiction if (X_1, \dots, X_L) is an absolutely continuous tuple.

If two or more X_i are dominant SU type, then (X_1, \dots, X_L) is automatically eligible. \square

4.2. Exceptional tuples that are not absolutely continuous.

LEMMA 2. *Suppose (X_1, \dots, X_L) is an exceptional tuple and is not a pair (X_1, X_2) of type $(SU(n), SU(n-1))$ in D_n where $n \geq 6$. Then (X_1, \dots, X_L) is not an absolutely continuous tuple.*

PROOF. We will need separate arguments for the various exceptional tuples.

(i) Suppose X_1 and X_2 are both type $SU(n)$ in the Lie algebra D_n . Observe that

$$\dim(sp\{Ad(g_i)(\mathcal{N}_{X_i}) : i = 1, 2\}) \leq |\mathcal{N}_{X_1}| + |\mathcal{N}_{X_2}|.$$

In this case, $|\mathcal{N}_{X_i}| = |\Phi_n|/2$. As the dimension of the Lie algebra is $|\Phi_n| + n$ it is clearly impossible for $sp\{Ad(g_i)(\mathcal{N}_{X_i}) : i = 1, 2\}$ to be the full Lie algebra. Thus Prop. 1(iii) proves that this pair is not absolutely continuous.

(ii) Suppose X_1 and X_2 are of types $SU(n)$ and $SU(n-1)$, respectively, in D_n with $n = 4$ or 5 . For this problem, we will use the fact that a root system of type $SU(4)$ is isomorphic to one of type D_3 . We will explain the argument for $n = 4$ and leave $n = 5$ as an exercise.

Let π be an automorphism of the root system of type D_4 (an isomorphism that preserves the Cartan matrix) that maps the annihilating roots of X_1 (those of type $SU(4)$) onto a root subsystem of type D_3 . This automorphism extends to an automorphism on the torus of D_4 which maps X_1 to the element $\pi(X_1)$ whose set of annihilating roots is the D_3 root subsystem, and it maps X_2 to the element $\pi(X_2)$ whose set of annihilating roots is isomorphic to those of X_2 and hence is type $SU(3)$ (as this is unique up to Lie isomorphism). It induces a Lie algebra isomorphism that we also call π . We have $\pi(O_{X_j}) = O_{\pi(X_j)}$ and

$$\pi(T_{Ad(g_j)(X_j)}(O_{X_j})) = T_{Ad(\pi(g_j))(\pi(X_j))}(O_{\pi(X_j)})$$

where if $g_j = \exp H_j$, then $\pi(g_j) = \exp \pi(H_j)$.

The pair $(\pi(X_1), \pi(X_2))$ is not eligible in D_4 as $S_{\pi(X_1)} = 6$ and $S_{\pi(X_2)} = 3$, so by our previous lemma it is not an absolutely continuous pair. Consequently, Prop. 1(iv) implies that

$$\dim \left(\sum_{i=1}^2 T_{Ad(\pi(g_i))\pi(X_i)}(O_{\pi(X_i)}) \right) < \dim D_n$$

for any choices of g_1, g_2 . But then a similar statement holds for $\sum_{i=1}^2 T_{Ad(g_i)X_j}(O_{X_j})$ and thus (X_1, X_2) is not an absolutely continuous pair.

(iii) When (X_1, X_2) is a pair of type $(SU(4), SU(2) \times D_2)$ in D_4 the arguments are similar. The Lie isomorphism, π , that maps the subsystem of type $SU(4)$ onto one of type D_3 must preserve the type of the root subsystem of type $SU(2) \times D_2$. But the pair $(\pi(X_1), \pi(X_2))$ is not eligible and hence neither it, nor the original pair, can be absolutely continuous.

Next, suppose X_1 is type $SU(4)$ and X_2 is type $SU(2) \times SU(2)$ in D_4 with the subsystem, Φ_{X_2} , Weyl conjugate to a subset of the subsystem Φ_{X_1} . Since any Weyl conjugate of X_2 generates the same orbit as X_2 there is no loss of generality in assuming $\Phi_{X_2} \subseteq \Phi_{X_1}$. Consider the same Lie isomorphism π again. Then $\pi(\Phi_{X_2}) \subseteq \pi(\Phi_{X_1})$ has the same Lie type as Φ_{X_2} . But the only subsystems of type D_3 that are isomorphic to type $SU(2) \times SU(2)$ are of the form $\{\pm e_i \pm e_j\}$ for some $i \neq j$, and hence are type D_2 . Being of type (D_3, D_2) , the pair $(\pi(X_1), \pi(X_2))$ is not eligible and therefore (X_1, X_2) is not absolutely continuous.

(iv) Assume X_1, X_2, X_3 are each of type $SU(4)$ in D_4 , with Weyl conjugate sets of annihilators. As the annihilators are Weyl conjugate, for each $i = 1, 2, 3$ there exist h_i in the Lie group of type D_4 such that $Ad(h_i)(\mathcal{N}_{X_1}) = \mathcal{N}_{X_i}$. Therefore there exist g_i in the group such that

$$sp\{Ad(g_i)(\mathcal{N}_{X_i}) : i = 1, 2, 3\} = \mathfrak{g}$$

if and only if

$$sp\{Ad(g_i h_i)(\mathcal{N}_{X_1}) : i = 1, 2, 3\} = \mathfrak{g}.$$

But the latter was shown to be impossible in the proof of [9, Thm. 8.2].

(v) The argument is similar if X_1 and X_2 are both type $SU(n) \times SU(n)$ in the Lie algebra of type $SU(2n)$. In this case, \mathcal{N}_{X_1} and \mathcal{N}_{X_2} are Weyl conjugate and it was shown in [9, Prop. 5.1] that there is no $g \in SU(2n)$ such that $sp\{Ad(g)\mathcal{N}_{X_1}, \mathcal{N}_{X_1}\} = su(2n)$. \square

5. Proving Absolute Continuity - Main Ideas

5.1. General Strategy. Our proof that the eligible, non-exceptional tuples are absolutely continuous will proceed by induction on the rank of the Lie algebra. The reduction is based upon the following idea.

NOTATION 2. Suppose X in the torus of the Lie algebra of type $SU(n)$, B_n , C_n or D_n is identified (after a suitable Weyl conjugate) with the n -vector

$$\underbrace{(0, \dots, 0)}_J, \underbrace{(a_1, \dots, a_1)}_{s_1}, \dots, \underbrace{(a_m, \dots, (\pm)a_m)}_{s_m}$$

where $s_1 = \max s_j$ and $J = 0$ in the case of type $SU(n)$. Define the element $X' \in \mathfrak{t}_{n-1}$ by

$$(5.1) \quad X' = \begin{cases} \underbrace{(0, \dots, 0)}_{J-1}, \underbrace{(a_1, \dots, a_1)}_{s_1}, \dots, \underbrace{(a_m, \dots, (\pm)a_m)}_{s_m} & \text{if } 2J \geq s_1 \\ \underbrace{(0, \dots, 0)}_J, \underbrace{(a_1, \dots, a_1)}_{s_1-1}, \dots, \underbrace{(a_m, \dots, (\pm)a_m)}_{s_m} & \text{if } 2J < s_1 \end{cases}$$

This means, for example, that if X has type $B_J \times SU(s_1) \times \dots \times SU(s_m)$ where $s_1 = \max s_j$, then X' has type $B_{J-1} \times SU(s_1) \times \dots \times SU(s_m)$ if X is dominant B type and X' has type $B_J \times SU(s_1 - 1) \times \dots \times SU(s_m)$ if X is dominant SU type. If X in $SU(n)$ has type $SU(s_1) \times \dots \times SU(s_m)$, then $S_{X'} = S_X - 1$ if $s_1 > \max_{j \geq 2} s_j$, and $S_{X'} = S_X$ otherwise. In the latter case $S_X \leq n/2$.

We can embed \mathfrak{t}_{n-1} into \mathfrak{t}_n by taking the standard basis vectors e_1, \dots, e_n in \mathbb{R}^n (or $e_1 - e_{n+1}, \dots, e_n - e_{n+1}$ in \mathbb{R}^{n+1} in the case of type $SU(n+1)$) as the basis for \mathfrak{t}_n and taking the vectors e_2, \dots, e_n (resp., $e_2 - e_{n+1}, \dots, e_n - e_{n+1}$) as the basis for \mathfrak{t}_{n-1} . This also gives a natural embedding of Φ_{n-1} into Φ_n and together these give an embedding of \mathfrak{g}_{n-1} into \mathfrak{g}_n , an embedding of \mathcal{V}_{n-1} into \mathcal{V}_n and an embedding of G_{n-1} into G_n . We will also view X' as an element of \mathfrak{t}_n in the natural way.

An induction argument will be applicable because of the following lemma.

LEMMA 3. *If (X, Y) is an eligible pair in \mathfrak{g}_n and X, Y are not both of type $SU(m) \times SU(m)$ in the Lie algebra of type $SU(2m)$, then the reduced pair, (X', Y') , is eligible in \mathfrak{g}_{n-1} .*

PROOF. Case 1: \mathfrak{g}_n is type B_n , C_n or D_n .

Observe that always $S_{X'} \leq S_X$ since the dimensions of the eigenspaces of X' can only be at most the dimensions of those of X .

If both X and X' are dominant B , C or D type, then $S_{X'} = S_X - 2$. If X' is dominant SU type, then $S_{X'} \leq n - 1$, regardless of the type of X . Finally, if X is dominant SU type while X' is dominant B , C or D type, then $S_X = s_1 > 2J = S_{X'} \geq s_1 - 1$. Since it is always true that $J + s_1 \leq n$, one can check that $s_1 \leq (2n + 1)/3$ and hence $S_{X'} \leq n - 1$.

Thus if either X and X' or Y and Y' are both dominant B , C or D type, then

$$S_{X'} + S_{Y'} \leq S_X + S_Y - 2 \leq 2(n - 1).$$

Otherwise, both $S_{X'}$ and $S_{Y'} \leq n-1$ and again we conclude that $S_{X'} + S_{Y'} \leq 2(n-1)$.

Case 2: \mathfrak{g}_n is type $SU(n+1)$.

If either $S_{X'} < S_X$ or $S_{Y'} < S_Y$, then $S_{X'} + S_{Y'} \leq S_X + S_Y - 1$ and thus (X', Y') is eligible. Otherwise, $S_{X'} = S_X$ and $S_{Y'} = S_Y$ and in that case $S_{X'}, S_{Y'} \leq (n+1)/2$. If n is even, then we must have $S_{X'}, S_{Y'} \leq n/2$ giving $S_{X'} + S_{Y'} \leq n$. If n is odd, it is still true that $S_{X'} + S_{Y'} \leq n$ unless $S_{X'} = S_{Y'} = (n+1)/2$. But that happens only when X and Y are both type $SU((n+1)/2) \times SU((n+1)/2)$, which is not permitted. \square

REMARK 4. It is easy to see that if X and X' are of opposite dominant types, then X is type B_J , $(C_J$ or $D_J) \times SU(s_1) \times \cdots \times SU(s_m)$ where $1 \leq J < \sum s_i$. It follows from [9, Thm. 8.2] that $\mu_X^2 \in L^2$.

We record here a well known fact from elementary linear algebra that is a consequence of the continuity of the determinant function and will be quite useful for us.

LEMMA 4. *If $\{v_1, \dots, v_n\}$ is a set of linearly independent vectors in vector space V and $w_1, \dots, w_n \in V$, then for sufficiently small $\varepsilon > 0$, the collection $\{v_1 + \varepsilon w_1, \dots, v_n + \varepsilon w_n\}$ is also linearly independent.*

NOTATION 3. Given X' as defined above, let $\mathcal{N}_{X'} = \{RE_\alpha, IE_\alpha : \alpha \notin \Phi_{X'}\}$, (as in 2.5), but viewed as embedded into \mathcal{V}_n . Let

$$\Omega_X = \mathcal{N}_X \setminus \mathcal{N}_{X'}.$$

We will refer to the next result as our general strategy. It will enable us to establish Prop. 1(iii) holds for a given tuple.

PROPOSITION 2. (*General Strategy*) *Let $X_i \in \mathfrak{t}_n$, $i = 1, \dots, L$ for $L \geq 2$, and assume (X'_1, \dots, X'_L) is an absolutely continuous tuple in \mathfrak{g}_{n-1} . Suppose Ω is a subset of $\mathcal{V}_n \setminus \mathcal{V}_{n-1}$ that contains all Ω_{X_i} and has the property that $\text{ad}(H)(\Omega) \subseteq \text{sp}\Omega$ whenever $H \in \mathfrak{g}_{n-1}$. Fix $\Omega_0 \subseteq \Omega_{X_L}$.*

Assume there exists $g_1, \dots, g_{L-1} \in G_{n-1}$ and $M \in \mathfrak{g}_n$ such that

(i) $\text{sp}\{Ad(g_i)(\Omega_{X_i}), \Omega_{X_L} \setminus \Omega_0 : i = 1, \dots, L-1\} = \text{sp}\Omega$;

(ii) $\text{ad}^k(M) : \mathcal{N}_{X_L} \setminus \Omega_0 \rightarrow \text{sp}\{\Omega, \mathfrak{g}_{n-1}\}$ for all positive integers k ; and

(iii) The span of the projection of $\text{Ad}(\exp sM)(\Omega_0)$ onto the orthogonal complement of $\text{sp}\{\mathfrak{g}_{n-1}, \Omega\}$ in \mathfrak{g}_n is a surjection for all small $s > 0$.

Then (X_1, \dots, X_L) is an absolute continuous tuple.

PROOF. As (X'_1, \dots, X'_L) is an absolutely continuous tuple, Prop. 1(iii) tells us that

$$\text{sp}\{Ad(h_i)(\mathcal{N}_{X'_i}), \mathcal{N}_{X'_L} : i = 1, \dots, L-1\} = \mathfrak{g}_{n-1}$$

for a dense set of $(h_1, \dots, h_{L-1}) \in G_{n-1}^{L-1}$. Given $\varepsilon > 0$, choose such $h_i = h_i(\varepsilon) \in G_{n-1}$ with $\|Ad(h_i) - Ad(g_i)\| < \varepsilon$, where the elements $g_i \in G_{n-1}$ are the ones given in the hypothesis of the proposition. (The norm can be taken to be the operator norm.)

Lemma 4, together with assumption (i), shows that for sufficiently small $\varepsilon > 0$,

$$\begin{aligned} \dim(\text{sp}\Omega) &= \dim(\text{sp}\{Ad(g_i)(\Omega_{X_i}), \Omega_{X_L} \setminus \Omega_0 : i = 1, \dots, L-1\}) \\ &= \dim(\text{sp}\{Ad(h_i)(\Omega_{X_i}), \Omega_{X_L} \setminus \Omega_0 : i = 1, \dots, L-1\}). \end{aligned}$$

Since $ad(H)(\Omega) \subseteq sp(\Omega)$ for all $H \in \mathfrak{g}_{n-1}$ and $h_i = \exp H_i$ for some $H_i \in \mathfrak{g}_{n-1}$ we have

$$\begin{aligned} Ad(h_i)(\Omega) &= Ad(\exp H_i)(\Omega) \\ &= \exp(ad(H_i)(\Omega)) \subseteq sp\Omega \end{aligned}$$

for all $h_i \in G_{n-1}$. Thus for sufficiently small $\varepsilon > 0$,

$$sp\{Ad(h_i)(\Omega_{X_i}), \Omega_{X_L} \setminus \Omega_0 : L = 1, \dots, L-1\} = sp\Omega$$

For such a choice of ε (hereafter fixed) we have

$$\begin{aligned} &sp\{Ad(h_i)(\mathcal{N}_{X_i}), \mathcal{N}_{X_L} \setminus \Omega_0 : i = 1, \dots, L-1\} \\ &= sp\{Ad(h_i)(\mathcal{N}_{X_i}), Ad(h_i)\Omega_{X_i}, \mathcal{N}_{X_L}, \Omega_{X_L} \setminus \Omega_0\} = sp\{\Omega, \mathfrak{g}_{n-1}\}. \end{aligned}$$

Assumption (ii), and the fact that $\mathcal{N}_{X_L} \setminus \Omega_0 \subseteq sp\{\Omega, \mathfrak{g}_{n-1}\}$, implies that for any real number s , $\exp(s \cdot adM) = Ad(\exp sM)$ maps $\mathcal{N}_{X_L} \setminus \Omega_0$ to $sp\{\Omega, \mathfrak{g}_{n-1}\}$. Moreover, $\|Id - Ad(\exp sM)\| \rightarrow 0$ as $s \rightarrow 0$, thus similar reasoning to that above shows that for all small enough $s > 0$,

$$\begin{aligned} sp\{\Omega, \mathfrak{g}_{n-1}\} &= sp\{Ad(h_i)(\mathcal{N}_{X_i}), \mathcal{N}_{X_L} \setminus \Omega_0 : i = 1, \dots, L\} \\ &= sp\{Ad(h_i)(\mathcal{N}_{X_i}), (Ad(\exp sM))(\mathcal{N}_{X_L} \setminus \Omega_0) : i = 1, \dots, L-1\}. \end{aligned}$$

Combined with assumption (iii), this proves that for sufficiently small $s > 0$,

$$sp\{Ad(h_i)(\mathcal{N}_{X_i}), Ad(\exp sM)(\mathcal{N}_{X_L}) : i = 1, \dots, L-1\} = \mathfrak{g}_n.$$

Another application of Prop. 1(iii) shows that $\mu_{X_1} * \dots * \mu_{X_L}$ is absolutely continuous. \square

We will occasionally make use of the following specific application of the elementary linear algebra property in order to verify the hypothesis of the general strategy.

LEMMA 5. *Suppose Ω is a subset of $\mathcal{V}_n \setminus \mathcal{V}_{n-1}$ that contains both Ω_X and Ω_Y , and has the property that $ad(H)(\Omega) \subseteq sp\Omega$ whenever $H \in \mathfrak{g}_{n-1}$. Fix $\Omega_0 \subseteq \Omega_X$. Assume $\Omega_1 \subseteq (\Omega_Y \cap \Omega_X) \setminus \Omega_0$ and the vectors in $\{adH(\Omega_1), \Omega_Y \setminus \Omega_1, \Omega_X \setminus \Omega_0\}$ span Ω for some $H \in \mathfrak{g}_{n-1}$. Then for sufficiently small $t > 0$,*

$$sp\{Ad(\exp tH)(\Omega_Y), \Omega_X \setminus \Omega_0\} = sp\Omega.$$

PROOF. The arguments are similar to that of the general strategy. Since $\|ad(H) - \frac{1}{t}(Ad(\exp tH) - Id)\|$ and $\|Id - Ad(\exp tH)\|$ both tend to 0 as $t \rightarrow 0$, and $ad^k(H)(\Omega) \subseteq sp\Omega$ for all k , the same argument as used above shows that

$$sp\{(Ad(\exp tH) - Id)(\Omega_1), Ad(\exp tH)(\Omega_Y \setminus \Omega_1), \Omega_X \setminus \Omega_0\} = sp\Omega.$$

But since $\Omega_1 \subseteq \Omega_X \setminus \Omega_0$, we can replace $(Ad(\exp tH) - Id)(\Omega_1)$ in the span on the left hand side by $Ad(\exp tH)(\Omega_1)$. Hence

$$sp\{Ad(\exp tH)(\Omega_Y), \Omega_X \setminus \Omega_0\} = sp\Omega.$$

\square

5.2. Applying the General strategy with $L = 2$. The following proposition, the ‘induction step’, is the most important ingredient in the proof of the main theorem.

We continue to use the notation $\Omega_X = \mathcal{N}_X \setminus \mathcal{N}_{X'}$, where X' was defined in (5.1).

PROPOSITION 3. *Suppose (X, Y) is an eligible pair in \mathfrak{g}_n other than X, Y both of type $SU(n)$ in D_n or type $SU(n/2) \times SU(n/2)$ in $SU(n)$. Assume also that the reduced pair, (X', Y') , is an absolutely continuous pair in \mathfrak{g}_{n-1} . Then (X, Y) is an absolutely continuous pair in \mathfrak{g}_n .*

PROOF. The main task of the proof is to show that any eligible pair, other than one of the two exceptional pairs mentioned, satisfy properties (i) - (iii) of the general strategy, Prop. 2.

Part I: \mathfrak{g}_n is type B_n, C_n or D_n .

The proof is divided into three cases depending on the dominant types of X and Y .

Case 1: Neither X nor Y are of dominant SU type.

With the notation as before, we have $S_X = 2J$ and $S_Y = 2K$ (meaning X is dominant B_J (C_J or D_J) type and Y is dominant B_K (C_K or D_K) type). Applying a Weyl conjugate, if necessary, we can assume without loss of generality that

$$\Omega_X = \{FEe_1 \pm e_j : J < j \leq n, F = R, I\}$$

and similarly

$$\Omega_Y = \{FEe_1 \pm e_j : K < j \leq n, F = R, I\}.$$

Case 1(a): \mathfrak{g}_n is type D_n .

Recall that \mathcal{V}_n is set of all real and imaginary parts of the chosen Weyl basis of root vectors of \mathfrak{g}_n . Put

$$\Omega = \mathcal{V}_n \setminus \mathcal{V}_{n-1} = \{FEe_1 \pm e_j : j = 2, \dots, n, F = R, I\}$$

and

$$\Omega_0 = \{REe_1 + e_n, IEe_1 + e_n\}.$$

If $H \in \mathfrak{g}_{n-1}$, then H is a linear combination of a torus element of \mathfrak{g}_{n-1} and the vectors $REe_i \pm e_j, IEe_i \pm e_j$ with $2 \leq i < j \leq n$. It follows easily from (2.4) that $ad(H)(\Omega) \subseteq sp\Omega$.

Take $g \in G_{n-1}$ to be the Weyl conjugate that permutes the letters $1+j$ and $K+j$ for $j = 1, \dots, J-1$. This is well defined and leaves the letter n unchanged as the eligibility condition ensures $J+K-1 \leq n-1$. Consequently, $Ad(g)(FEe_1 \pm e_{K+j}) = FEe_1 \pm e_{1+j}$ for $j = 1, \dots, J-1$, and all other vectors in Ω are fixed, including $FEe_1 \pm e_n$. Thus

$$\{Ad(g)(\Omega_Y), \Omega_X \setminus \Omega_0\} = \{FEe_1 \pm e_k : k = 2, \dots, n\}$$

proving that (i) of the general strategy, Prop 2 (with $L = 2$) is satisfied.

Let $M = REe_1 + e_n \in \mathfrak{g}_n$. Applying (2.4) again, we see that if $H = FEe_1 \pm e_j$ for some $j < n$, then $ad(M)(H) = cFEe_j \mp e_n \in \mathfrak{g}_{n-1}$ for a non-zero constant c depending on j, n and F . If $H = FEe_i \pm e_n$, then $ad(M)(H) = cFEe_1 \mp e_i \in sp(\Omega \setminus \Omega_0)$. Finally, note that $ad(M)(H) = 0$ if $H = FEe_i \pm e_j$ for $1 < i, j < n$ or $H = FEe_1 - e_n$. This proves $ad^k(M) : \mathcal{N}_X \setminus \Omega_0 \rightarrow sp\{\Omega, \mathfrak{g}_{n-1}\}$ for all positive integers k , so that property (ii) of the general strategy is satisfied.

As $sp\{\Omega, \mathfrak{g}_{n-1}\}$ is of co-dimension one, its orthogonal complement is spanned by the projection onto any element in the complement of $sp\{\Omega, \mathfrak{g}_{n-1}\}$. The torus element,

$$ad(M)(IEe_1 + e_n) = [REe_1 + e_n, IEe_1 + e_n] := t_1$$

is such an element. Since

$$ad(M)(t_1) = [REe_1 + e_n, t_1] = cIEe_1 + e_n$$

for some $c \neq 0$ (see 2.3), it follows that

$$Ad(\exp sM)(IEe_1 + e_n) = a(s)IEe_1 + e_n + sb(s)t_1$$

where $a(s), b(s) \rightarrow 1$ as $s \rightarrow 0$. Therefore hypothesis (iii) of Prop. 2 is also fulfilled with any $s > 0$. Applying that proposition, we conclude that $\mu_X * \mu_Y$ is absolutely continuous.

Case 1(b): g_n is type B_n .

Again, we will apply the general strategy, but here with

$$\Omega = \mathcal{V}_n \setminus \mathcal{V}_{n-1} = \{FEe_1 \pm e_j, FEe_1 : j = 2, \dots, n, F = R, I\}$$

and

$$\Omega_0 = \{REe_1 + e_n, IEe_1 + e_n\}.$$

The fact that $ad(H)(\Omega) \subseteq \Omega$ whenever $H \in \mathfrak{g}_{n-1}$ follows easily from properties of the roots, as with the case D_n .

For $t > 0$, let $g_t = (\exp tREe_n)g$ where $g \in G_{n-1}$ corresponds to the Weyl conjugate that permutes the letters $1+j$ and $K+j$ for $j = 1, \dots, J-1$ as in the previous case. Since $REe_n \in \mathfrak{g}_{n-1}$, $g_t \in G_{n-1}$. Observe that

$$[REe_n, FEe_1] = cFEe_1 + e_n + c'FEe_1 - e_n$$

and

$$[REe_n, FEe_1 \pm e_j] = \begin{cases} c^{(\pm)}FEe_1 & \text{if } j = n \\ 0 & \text{else} \end{cases}$$

with c, c' and $c^{(\pm)}$ non-zero constants. In particular, this implies

$$Ad(\exp tREe_n)(FEe_1 \pm e_j) = FEe_1 \pm e_j \text{ for } j \neq n.$$

Since $Ad(g)(FEe_1 \pm e_{K+j}) = FEe_1 \pm e_{1+j}$ for $j = 1, \dots, J-1$ and the eligibility condition ensures $Ad(g)$ fixes $FEe_1 \pm e_n$, it follows that for $j = 1, \dots, J-1$ we have

$$\begin{aligned} Ad(g_t)(FEe_1 \pm e_{K+j}) &= Ad(\exp tREe_n)(FEe_1 \pm e_{1+j}) \\ &= FEe_1 \pm e_{1+j} \end{aligned}$$

and

$$\begin{aligned} Ad(g_t)(FEe_1 \pm e_n) &= Ad(\exp tREe_n)(FEe_1 \pm e_n) \\ &= a(t)FEe_1 \pm e_n + tb(t)FEe_1 + t^2c(t)FEe_1 \mp e_n \end{aligned}$$

where $a(t) \rightarrow 1$ as $t \rightarrow 0$, and $b(t)$ and $c(t)$ converge to non-zero scalars². All other choices of $FEe_1 \pm e_j$ are fixed by $Ad(g_t)$. Hence

$$\begin{aligned} &sp\{FEe_1 - e_n, Ad(g_t)(FEe_1 \pm e_n) : F = R, I\} \\ &= sp\{FEe_1 - e_n, FEe_1 + e_n + tb'(t)FEe_1, FEe_1 + tc'(t)FEe_1 + e_n : F = R, I\} \end{aligned}$$

² $a(t), b(t), c(t)$ depend on F and the choice of \pm , as well as t . From here on we will omit noting this dependence, unless it is important.

where $b'(t)$ and $c'(t)$ converge to non-zero limits as $t \rightarrow 0$. Since

$$\{FEe_1 \pm e_n, FEe_1 : F = R, I\}$$

is a set of six linearly independent vectors, so too is the collection

$$\{FEe_1 - e_n, FEe_1 + e_n + tb'(t)FEe_1, FEe_1 + tc'(t)FEe_1 + e_n : F = R, I\}$$

for sufficiently small t , and therefore they span the same space. Because $\Omega_X \setminus \Omega_0$ contains $FEe_1 - e_n$, it follows that

$$\begin{aligned} & sp\{Ad(g_t)(\Omega_Y), \Omega_X \setminus \Omega_0\} \\ &= sp\{Ad(g_t)(FEe_1 \pm e_k), FEe_1 \pm e_j, FEe_1 - e_n : k > K, J < j < n, F = R, I\} \\ &= sp\{FEe_1 \pm e_j, FEe_1 \pm e_n, FEe_1 : j \leq n, F = R, I\} = sp\Omega. \end{aligned}$$

Again, put $M = REe_1 + e_n \in \mathfrak{g}_n$. As with type D_n , $ad^k(M)(FEe_1 \pm e_j) \in sp\{\mathfrak{g}_{n-1}, \Omega\}$ for all k and $j < n$, and $ad(M)(FEe_1 - e_n) = 0$. Furthermore, $ad(M)(FEe_j) = 0$ if $j \neq 1, n$, $ad(M)(FEe_n) = cFEe_1$ and $ad(M)(FEe_1) = cFEe_n$, so property (ii) of the general strategy holds. As in the first case, $sp\{\mathfrak{g}_{n-1}, \Omega\}$ is of co-dimension one in \mathfrak{g}_n , and just as in type D_n property (iii) holds, so we deduce the absolute continuity of $\mu_X * \mu_Y$ by appealing to Prop. 2.

Case 1(c): g_n is type C_n .

Here we will use a variant on the general strategy. As with type D_n we begin with

$$\Omega = \{FEe_1 \pm e_j : j = 2, \dots, n, F = R, I\}$$

and g the Weyl conjugate permuting the letters $1+j$ and $K+j$ for $j = 1, \dots, J-1$. Take

$$\Omega_0 = \{FEe_1 \pm e_n : F = R, I\}.$$

The eligibility condition gives that $sp\{Ad(g)(\Omega_Y), \Omega_X \setminus \Omega_0\} = sp\Omega$.

As with type D_n , $ad(FEe_i \pm e_j)(\Omega) \subseteq sp\{\Omega, \mathfrak{g}_{n-1}\}$ for all $1 < i < j \leq n$ and similarly, $ad(FE(2e_j))(\Omega) \subseteq \Omega$ for $j > 1$, so $ad(H)(\Omega) \subseteq sp\{\Omega, \mathfrak{g}_{n-1}\}$ whenever $H \in \mathfrak{g}_{n-1}$. Thus, as in the proof of the general strategy, upon applying the induction assumption we can deduce there is some $h \in G_{n-1}$ such that

$$(5.2) \quad sp\{Ad(h)(\mathcal{N}_Y), \mathcal{N}_X \setminus \Omega_0\} = sp\{\Omega, \mathfrak{g}_{n-1}\}.$$

Once again, we will put $M = REe_1 + e_n \in \mathfrak{g}_n$. As with the types B_n and D_n , standard facts about roots show that $ad(M)(H) \in sp\{\Omega, \mathfrak{g}_{n-1}\}$ for all $H \in \mathcal{N}_X \setminus \Omega_0$. In fact, for all $k \geq 1$, $ad^k(M)(H) \in sp\{\Omega, \mathfrak{g}_{n-1}\}$ for all $H \in \mathcal{N}_X \setminus \Omega_0$ except for $H = FE(2e_n)$ as $ad^k(M)(FE(2e_n))$ has a component in $FE(2e_1)$. (Recall that $FE(2e_n) \in \mathcal{N}_X$ since the only roots $2e_j \in \Phi_X$ are those with $j \leq J$.) It is because of this exception that we cannot appeal directly to the general strategy.

Another difference between this set up and the situation for types B_n and D_n is that here $sp\{\Omega, \mathfrak{g}_{n-1}\}$ has co-dimension three, its orthogonal complement being spanned by $RE(2e_1), IE(2e_1)$ and the projection onto the torus element $[REe_1 + e_n, IEe_1 + e_n]$. That will also complicate matters.

Let Λ be the subspace spanned by the torus of \mathfrak{g}_{n-1} and the vectors RE_β and IE_β where β ranges over all the positive roots except $2e_1, 2e_n$,

$$\Lambda := sp\{\Omega, \mathfrak{g}_{n-1}\} \ominus sp\{RE(2e_n), IE(2e_n)\}.$$

Let \mathcal{P} be the orthogonal projection onto Λ . Since $\mathcal{N}_X \setminus \{\Omega_0, FE(2e_n)\} \subseteq \Lambda$, Property (5.2) implies that

$$sp\{\mathcal{P}(Ad(h)(\mathcal{N}_Y)), \mathcal{N}_X \setminus \{\Omega_0, RE(2e_n), IE(2e_n)\}\} = \Lambda.$$

Choose $Y_\beta^F, Y_j \in Ad(h)\mathcal{N}_Y$ and $X_\beta^F, X_j \in \mathcal{N}_X \setminus \{\Omega_0, RE(2e_n), IE(2e_n)\}$ such that

- (i) $Y_\beta^F + X_\beta^F = FE_\beta + W_\beta^F$ where $W_\beta^F \in sp\{RE(2e_n), IE(2e_n)\}$, $F = R, I$ and β ranges over all roots except $2e_1, 2e_n$, and
- (ii) $Y_j + X_j = t_j + W_j$ where $j = 2, \dots, n$, $\{t_2, \dots, t_n\}$ is a basis for \mathfrak{t}_{n-1} and $W_j \in sp\{FE(2e_n)\}$.

Note that if we put $t_1 = [REe_1 + e_n, IEe_1 + e_n]$, then $\{t_1, \dots, t_n\}$ is a basis for \mathfrak{t}_n .

This collection of vectors $\{Y_\beta^F + X_\beta^F, Y_j + X_j\}$ is linearly independent and hence for small enough $s > 0$, so is also the set

$$\{Y_\beta^F + Ad(\exp sM)(X_\beta^F), Y_j + Ad(\exp sM)(X_j) : \beta \neq 2e_1, 2e_n, j = 2, \dots, n, F = R, I\}.$$

Observe that

$$\begin{aligned} Y_\beta^F + Ad(\exp sM)(X_\beta^F) &= Y_\beta^F + X_\beta^F + (Ad(\exp sM) - Id)(X_\beta^F) \\ &= FE_\beta + W_\beta^F + sQ_\beta^F(s), \end{aligned}$$

where the vector $Q_\beta^F(s)$ depends on s , but has bounded norm. The projection of $Q_\beta^F(s)$ onto $sp\{RE2e_1, IE2e_1\}$ is zero since $Ad(\exp sM)$ maps $\mathcal{N}_X \setminus \{\Omega_0, FE(2e_n) : F = R, I\}$ into $\mathfrak{g}_n \ominus sp\{RE2e_1, IE2e_1\}$. Also, it is clear from the definitions that for $\beta \neq e_1 - e_n$, the projection of $FE_\beta + W_\beta^F$ onto $sp\{REe_1 - e_n, IEe_1 - e_n\}$ is zero. Similar statements can be made for $Y_j + Ad(\exp sM)(X_j)$.

Claim: The collection of vectors, $Y_\beta^F + Ad(\exp sM)(X_\beta^F)$, $Y_j + Ad(\exp sM)(X_j)$ over all positive roots $\beta \neq 2e_1, 2e_n$, $F = R, I$, and $j = 2, \dots, n$, together with the four vectors $Ad(\exp sM)(FE(2e_n))$, $Ad(\exp sM)(FEe_1 - e_n)$ for $F = R, I$, are linearly independent.

To prove this we first observe that

$$\begin{aligned} [REe_1 + e_n, FE(2e_1)] &= c_1 FEe_1 - e_n \\ [REe_1 + e_n, FE(2e_n)] &= c_2 FEe_n - e_1, \\ [REe_1 + e_n, FEe_1 - e_n] &= c_3 FE(2e_1) + c_4 FE(2e_n) \end{aligned}$$

where $c_j \neq 0$. Thus

$$(5.3) \quad Ad(\exp sM)(FEe_1 - e_n) = a_s^F FEe_1 - e_n + sb_s^F FE(2e_1) + sc_s^F FE(2e_n)$$

and

$$(5.4) \quad Ad(\exp sM)(FE(2e_n)) = sb_s'^F FEe_1 - e_n + s^2 c_s'^F FE(2e_1) + a_s'^F FE(2e_n)$$

where the coefficients, $a_s^F, a_s'^F, b_s^F, b_s'^F, c_s^F, c_s'^F$, converge to non-zero constants as $s \rightarrow 0$.

The vectors listed in (5.3) and (5.4), as well as those in $sp\{\mathfrak{g}_{n-1}, \Omega\}$, belong to $\mathfrak{g}_n \ominus sp\{t_1\}$. We view them as vectors in \mathbb{R}^d with $d = \dim \mathfrak{g}_n - 1$, whose coordinates are given by the basis for $\mathfrak{g}_n \ominus sp\{t_1\}$ consisting of the torus elements, $\{t_2, \dots, t_n\}$, together with the real and imaginary parts of the Weyl basis $\{E_\alpha\}$, taking as the final six positions the basis vectors $FEe_1 - e_n$, $FE(2e_n)$ and $FE(2e_1)$, $F = R, I$.

With this understanding, consider the square matrix whose rows are given by the vectors $Y_j + Ad(\exp sM)(X_j)$ for $j = 2, \dots, n$; followed by the vectors

$Y_\beta^F + Ad(\exp sM)(X_\beta^F)$, $\beta \neq 2e_1, 2e_n$, ordered consistently to above so that the final two come from $\beta = e_1 - e_n$; and then finally the four vectors $Ad(\exp sM)(FE(2e_n))$ and $Ad(\exp sM)(FEe_1 - e_n)$ (for a small, but fixed, choice of s).

The calculations above show that this matrix, denoted $A = (A_{ij})$, has the form

$$A = \begin{bmatrix} [I_{d-6} + O(s)]_{(d-6) \times (d-6)} & [O(s)]_{(d-6) \times 2} & [*]_{(d-6) \times 2} & [0]_{(d-6) \times 2} \\ [O(s)]_{2 \times (d-6)} & [I_2 + O(s)]_{2 \times 2} & \begin{bmatrix} * & * \\ * & * \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ [0]_{2 \times (d-6)} & \begin{bmatrix} sb_s^R & 0 \\ 0 & sb_s^I \end{bmatrix} & \begin{bmatrix} a_s^R & 0 \\ 0 & a_s^I \end{bmatrix} & \begin{bmatrix} O(s^2) & 0 \\ 0 & O(s^2) \end{bmatrix} \\ [0]_{2 \times (d-6)} & \begin{bmatrix} a_s^R & 0 \\ 0 & a_s^I \end{bmatrix} & \begin{bmatrix} sc_s^R & 0 \\ 0 & sc_s^I \end{bmatrix} & \begin{bmatrix} sb_s^R & 0 \\ 0 & sb_s^I \end{bmatrix} \end{bmatrix}$$

where I_m denotes the $m \times m$ identity matrix, $O(s^k)$ means terms dominated by Cs^k for some constant C independent of s and $*$ denotes terms that may depend on s , but are bounded independently of s .

We estimate the determinant of this matrix using the Leibniz formula: Since $|A_{11}A_{22} \cdots A_{dd}| \geq C_0 s^2$ for some $C_0 > 0$ and all the other products $A_{1\sigma(1)}A_{2\sigma(2)} \cdots A_{d\sigma(d)}$, where σ is a permutation of $\{1, \dots, d\}$, are dominated in absolute value by $C_1 s^3$, the determinant is non-zero for sufficiently small $s > 0$. This completes the proof of the claim.

As there are the appropriate number of vectors, these vectors form a basis for $\mathfrak{g}_n \ominus sp\{t_1\}$. Recall that X_β^F , $FE(2e_n)$ and $FEe_1 - e_n$ all belong to $\mathcal{N}_X \setminus \{FEe_1 + e_n : F = R, I\}$, hence

$$sp\{Ad(h)\mathcal{N}_Y, Ad(\exp sM)(\mathcal{N}_X \setminus \{FEe_1 + e_n\})\} = \mathfrak{g}_n \ominus sp\{t_1\}.$$

Finally, our familiar calculation shows

$$Ad(\exp sM)(IEe_1 + e_n) = a_s IEe_1 + e_n + sb_s t_1$$

where b_s converges to a non-zero constant. It follows that for small enough s ,

$$sp\{Ad(h)(\mathcal{N}_Y), Ad(\exp sM)(\mathcal{N}_X)\} = \mathfrak{g}_n,$$

as we desired to show.

Case 2: Both X and Y are dominant SU type.

First, assume the Lie algebra is type B_n or C_n . According to [9, Thm. 8.2] both μ_X^2 and μ_Y^2 belong to L^2 . Applying Holder's inequality we see that $\mu_X * \mu_Y \in L^2$. Being compactly supported, it follows that $\mu_X * \mu_Y$ is in L^1 , and hence is a measure that is absolutely continuous with respect to Lebesgue measure. (Note that the same argument applies to the Lie algebra of type D_n unless one of X or Y is of type $SU(n)$.)

However, we prefer to give an argument that is independent of [9] as the techniques will then have more general application and such an argument will be needed in the case of type D_n , in any case. For this, in the case of type B_n , put

$$\Omega = \{FEe_1, FEe_1 \pm e_j : j \geq 2, F = R, I\} \text{ and}$$

$$\Omega_0 = \{REe_1, IEe_1\}$$

while in the case of type C_n , put

$$\Omega = \{FE(2e_1), FFe_1 \pm e_j : j \geq 2, F = R, I\} \text{ and} \\ \Omega_0 = \{RE(2e_1), IE(2e_1)\}.$$

In either case $ad(H)(\Omega) \subseteq sp\Omega$ for all $H \in \mathfrak{g}_{n-1}$.

As X, Y are dominant SU type, both Ω_X and Ω_Y contain $FE(2)e_1$ and all the roots $FEe_1 + e_j$, $j \geq 2$. If $g \in G_{n-1}$ is the Weyl conjugate that changes the signs of the letters $2, \dots, n$, then $\{Ad(g)(\Omega_X), \Omega_Y \setminus \Omega_0\} = \Omega$. Now take $M = RE(2)e_1$ and apply the general strategy.

The arguments are similar when the Lie algebra is type D_n . Let

$$\Omega = \{FEe_1 \pm e_j : j \geq 2, F = R, I\}.$$

As we do not permit both X and Y to be of type $SU(n)$, without loss of generality Ω_X contains all the roots $FEe_1 + e_j$ for $2 \leq j \leq n-1$, as well as both $FEe_1 \pm e_n$, and Ω_Y contains either all $FEe_1 + e_j$ for $2 \leq j$ or all $FEe_1 + e_j$ for $2 \leq j \leq n-1$ and $FEe_1 - e_n$. Let Ω_0 be the choice of $\{FEe_1 + e_n\}$ or $\{FEe_1 - e_n\}$, depending on which belongs to Ω_Y . Let $g \in G_{n-1}$ be the Weyl conjugate that changes the signs $2, \dots, n-1$ (and n if needed to be an even sign change). Then $Ad(g)(\Omega_X) \supseteq \{FEe_1 - e_j, FFe_1 \pm e_n : j \geq 2\}$ and hence $\{Ad(g)(\Omega_X), \Omega_Y \setminus \Omega_0\} = \Omega$. Take $M = REe_1 \pm e_n$ with the choice of \pm depending on which belongs to Ω_Y .

Case 3: X and Y are of different dominant type.

Without loss of generality assume X is dominant $SU(m)$ type and Y is dominant B_J, C_J or D_J type, depending on the type of the Lie algebra. Eligibility implies that $2J + m \leq 2n$.

Let

$$\Omega = \{FEe_1 \pm e_j, (FE(2)e_1) : j \geq 2\}.$$

(with the inclusion of FEe_1 if the Lie algebra is type B_n or $FE(2e_1)$ if the Lie algebra is C_n). We have

$$\Omega_X = \{FEe_1 + e_j, FFe_1 - e_n : j < n, \} \text{ if } X = (a, \dots, a, -a) \text{ in } D_n$$

and

$$\Omega_X = \{FEe_1 + e_j, FFe_1 - e_k, (FE(2)e_1) : j \geq 2, k > m, F = R, I\} \text{ otherwise.}$$

Put

$$\Omega_0 = \{FEe_1 + e_{n-J+1}\} \subseteq \Omega_X \cap \Omega_Y$$

(or $\Omega_0 = \{FEe_1 - e_n\}$ if $J = 1$ and $X = (a, \dots, a, -a)$ in D_n). Applying a Weyl conjugate from G_{n-1} , we can assume

$$\Omega_Y = \{FEe_1 \pm e_j : 2 \leq j \leq n - J + 1, F = R, I\}.$$

If $n - J + 1 \geq m$, then we already have

$$\{\Omega_Y, \Omega_X \setminus \Omega_0\} = \Omega,$$

so property (i) of the general strategy holds with $g = Id$. Take $M = REe_1 + e_{n-J+1}$ (resp., take $M = REe_1 - e_n$) to complete the argument.

Otherwise $m + J - n \geq 2$ (which implies $J \geq 2$). Put

$$\Omega_1 = \{FEe_1 + e_k : 2 \leq k \leq n - J, F = R, I\} \subseteq (\Omega_Y \cap \Omega_X) \setminus \Omega_0$$

and define

$$H = \sum_{j=2}^{J-1} REe_j + e_{n-J+j} + REe_{J-e_n} \text{ if } X = (a, \dots, a, -a) \text{ in type } D_n$$

and

$$H = \sum_{j=2}^{m+J-n} REe_j + e_{n-J+j} \text{ otherwise.}$$

As $J \neq n$, $e_j + e_{n-J+j}$ are roots of the Lie algebra \mathfrak{g}_{n-1} . Let $2 \leq k \leq m+J-n$. Observe that $k \neq n-J+j$ for any $j \geq 2$, for if so, then $j = k-n+J \leq m+2J-2n$ and therefore the eligibility condition would imply $j \leq 0$. Thus, if $2 \leq k \leq m+J-n$, then $ad(H)(FEe_1 + e_k) = c_k FEe_1 - e_{n-J+k}$ (or $ad(H)(FEe_1 + e_J) = c_J FEe_1 + e_n$ if $X = (a, \dots, -a)$).

The eligibility condition also implies

$$\Omega_1 \supseteq \{FEe_1 + e_k : 2 \leq k \leq m+J-n\},$$

therefore

$$sp\{ad(H)(\Omega_1)\} \supseteq sp\{FEe_1 - e_j, FEe_1 + e_n : n-J+2 \leq j \leq n-1, F = R, I\}$$

if $X = (a, \dots, a, -a)$ in type D_n and

$$sp\{ad(H)(\Omega_1)\} \supseteq sp\{FEe_1 - e_j : n-J+2 \leq j \leq m, F = R, I\} \text{ otherwise.}$$

Since $\Omega_Y \setminus \Omega_1 = \{FEe_1 - e_j, FEe_1 + e_{n-J+1} : 2 \leq j \leq n-J+1\}$, in either case we have

$$sp\{ad(H)(\Omega_1), \Omega_Y \setminus \Omega_1, \Omega_X \setminus \Omega_0\} = sp\Omega.$$

By Lemma 5 there is some $g \in G_{n-1}$ (namely, $g = \exp tH$ for sufficiently small t) such that

$$sp\{Ad(g)(\Omega_Y), \Omega_X \setminus \Omega_0\} = sp\Omega.$$

Again, take $M = REe_1 + e_{n-J+1}$ and apply the general strategy to complete the argument.

Part II: g_n is type $SU(n)$.

This is very similar to case 1(a). Let

$$\Omega = \{FEe_1 - e_j : 2 < j \leq n, F = R, I\}.$$

We have

$$\begin{aligned} \Omega_X &= \{FEe_1 - e_j : S_X < j \leq n, F = R, I\} \text{ and} \\ \Omega_Y &= \{FEe_1 - e_j : S_Y < j \leq n, F = R, I\}. \end{aligned}$$

Put $\Omega_0 = \{FEe_1 - e_n : F = R, I\}$. Take $g \in SU(n-1)$ to be the Weyl conjugate that interchanges the letters $S_Y + j$ and $1 + j$ for $j = 1, \dots, S_X - 1$. The eligibility condition ensures this is well defined and leaves 1 and n unchanged. Clearly $\{Ad(g)(\Omega_Y), \Omega_X \setminus \Omega_0\} = \Omega$. Take $M = REe_1 - e_n$ and apply the general strategy in the usual manner. \square

6. Proof of the Main Theorem

In this section we will complete the proof of Theorem 1.

PROOF OF THEOREM 1. Necessary conditions for Absolute continuity:

Lemma 1 shows that absolutely continuous tuples are eligible, while in Lemma 2 we saw that the exceptional tuples, other than possibly the pairs of type $(SU(n), SU(n-1))$ in the Lie algebra of type D_n with $n \geq 6$, are not absolutely continuous.

The rest of the proof is devoted to establishing that the eligible, non-exceptional tuples are absolutely continuous.

Sufficient conditions for Absolute continuity for Lie types A_n , B_n and C_n :

Case $L = 2$. The proof proceeds by induction on the rank n of the Lie algebra. We begin A_n with $n = 1$ (type $SU(2)$) and B_n with $n = 2$. Although it is customary to only define C_n for $n \geq 3$, there is no harm in beginning with C_2 , meaning the root system $\pm\{2e_1, 2e_2, e_1 \pm e_2\}$, which is Lie isomorphic to B_2 .

According to [9, Thm. 8.2], all non-zero pairs (X, Y) in the Lie algebras of type $SU(2)$ and B_2 have the property that both $\mu_X^2, \mu_Y^2 \in L^2$. Thus $\mu_X * \mu_Y$ is a compactly supported measure in L^2 and hence is an absolutely continuous measure. The existence of $g_1, g_2 \in G_n$ with

$$\sum_{i=1}^2 T_{Ad(g_i)(X_i)}(O_{X_i}) = \mathfrak{g}_n$$

is a Lie isomorphism invariant, thus from Prop. 1 we can also deduce that $\mu_X * \mu_Y$ is an absolutely continuous measure for all non-zero (X_1, X_2) in the Lie algebra of type C_2 .

Now, inductively assume that all eligible, non-exceptional pairs in $SU(n-1)$, B_{n-1} or C_{n-1} , with $n \geq 3$, are absolutely continuous. (Of course, there are no exceptional pairs in B_{n-1} or C_{n-1} .)

Let (X, Y) be an eligible, non-exceptional pair in $SU(n)$, B_n or C_n , and form the reduced pair (X', Y') . The reduced pair is eligible by Lemma 3. Notice that only an element of type $SU(\frac{n+1}{2}) \times SU(\frac{n-1}{2})$ in $SU(n)$ will reduce to an element of type $SU(\frac{n-1}{2}) \times SU(\frac{n-1}{2})$ in $SU(n-1)$. Furthermore, a pair of elements each of type $SU(\frac{n+1}{2}) \times SU(\frac{n-1}{2})$ is not eligible in $SU(n)$, thus we can assume (X', Y') is both eligible and non-exceptional. By the induction assumption, (X', Y') is an absolutely continuous pair. But then the induction step, Prop. 3, implies that (X, Y) is absolutely continuous.

Case $L \geq 3$. Again, we proceed by induction on n . We remark that as $\mu * \nu$ is absolutely continuous if μ is absolutely continuous and ν is an arbitrary measure, the fact that the convolution of any two non-zero orbital measures in type $SU(2)$, B_2 or C_2 is absolutely continuous, proves that the same is true for the convolution of any L non-zero orbital measures. This starts the induction.

First, suppose (X_1, \dots, X_L) is an eligible L -tuple in B_n or C_n with $n \geq 3$. We will let Ω be as in Prop. 3, depending on whether \mathfrak{g} is type B_n or C_n ,

$$\Omega = \{FEe_1 \pm e_j, FE(2)e_1 : j = 2, \dots, n, F = R, I\}.$$

As a pair of elements that is dominant SU type in B_n or C_n is eligible and not exceptional, the theorem for $L = 2$ implies the convolution of (even) their two

orbital measures is absolutely continuous. Thus we may assume that at most one X_i is dominant SU type.

Suppose that no X_i are dominant SU type and form the corresponding X'_i . If X'_i and X'_j are dominant SU , then the pair (X_i, X_j) is eligible (and not exceptional), thus $\mu_{X_i} * \mu_{X_j}$ is absolutely continuous. Hence we can assume that at most one X'_i is dominant SU type.

Since $S_{X'} = S_X - 2$ when both X and X' are dominant B (or C) type it follows that

$$\sum_{i=1}^L S_{X'_i} \leq \sum_{i=1}^L S_{X_i} - 2(L-1) \leq 2n(L-1) - 2(L-1) = 2(n-1)(L-1).$$

This shows that (X'_1, \dots, X'_L) is eligible in \mathfrak{g}_{n-1} . As it is not exceptional, the induction assumption implies it is an absolutely continuous tuple.

Here $\Omega_{X_i} = \{FEe_1 \pm e_j : j > J_i\}$ where $2J_i = S_{X_i}$. Taking g_i to be the Weyl conjugate that switches appropriate letters (and fixes the letters 1 and n) we can arrange for

$$Ad(g_i)(\Omega_{X_i}) = \{FEe_1 \pm e_j : j = (i-1)n - \sum_{k=1}^{i-1} J_k + 2, \dots, in - \sum_{k=1}^i J_k + 1\}$$

(with suitable modifications if any of the specified choices of j exceed n).

If $(L-1)n - \sum_{k=1}^{L-1} J_k + 1 \geq n$, then

$$\bigcup_{i=1}^{L-1} Ad(g_i)(\Omega_{X_i}) = \{FEe_1 \pm e_j : j = 2, \dots, n\},$$

and this coincides with the set Ω_Y for a suitable Y of type B_1 (or C_1) (meaning type B_1 (or C_1) $\times SU(1) \times \dots \times SU(1)$). As always $S_X \leq 2(n-1)$, the pair (Y, X_L) is eligible.

Otherwise, if we let $m = n - (L-1)n + \sum_{i=1}^{L-1} J_i$ and take a suitable choice of Y of type B_m (or C_m), then

$$\bigcup_{i=1}^{L-1} Ad(g_i)\Omega_{X_i} = \Omega_Y.$$

The eligibility condition ensures that

$$\begin{aligned} S_Y + S_{X_L} &= 2 \left(n - (L-1)n + \sum_{i=1}^{L-1} J_i \right) + 2J_L \\ &\leq 2n - 2(L-1)n + \sum_{i=1}^L S_{X_i} \leq 2n \end{aligned}$$

and thus the pair (Y, X_L) is eligible and clearly not exceptional. The arguments given in the proof of Prop. 3 Case 1 show that then there is some $g \in G_{n-1}$, $M \in \mathfrak{g}_n$ and $\Omega_0 \subseteq \Omega_{X_L}$ such that

- (i) $sp\Omega = sp\{Ad(g)(\Omega_Y), \Omega_{X_L} \setminus \Omega_0\}$;
- (ii) $ad^k(M) : \mathcal{N}_{X_L} \setminus \Omega_0 \rightarrow sp\{\Omega, \mathfrak{g}_{n-1}\}$ for all positive integers k ; and
- (iii) The span of the projection of $Ad(\exp sM)(\Omega_0)$ onto the orthogonal complement of $sp\{\mathfrak{g}_{n-1}, \Omega\}$ in \mathfrak{g}_n is a surjection for all small $s > 0$.

Since

$$sp\{Ad(g)(\Omega_Y), \Omega_{X_L} \setminus \Omega_0\} \subseteq sp\{Ad(gg_i)(\Omega_{X_i}), \Omega_{X_L} \setminus \Omega_0 : i = 1, \dots, L-1\}$$

we can call upon the general strategy, Prop. 2, with g_i replaced there by gg_i , to deduce that (X_1, \dots, X_L) is an absolutely continuous tuple. This completes the argument when no X_i are of dominant SU type.

Otherwise, there is one X_i which is of dominant SU type, say X_L . If there is another index j such that X'_j is of dominant SU type, then the pair (X_L, X_j) is eligible and not exceptional, hence $\mu_{X_L} * \mu_{X_j}$ is absolutely continuous.

So we may assume all X'_j with $j \neq L$ are of dominant B (or C) type. Thus

$$\sum S'_{X_i} \leq \sum S_{X_i} - 2(L-1) \leq 2(n-1)(L-1),$$

so (X'_1, \dots, X'_L) is an eligible L -tuple. Again, taking g_i to be suitable Weyl conjugates we have

$$\bigcup_{i=1}^{L-1} Ad(g_i)(\Omega_{X_i}) = \{FEE_1 \pm e_j : j = 2, \dots, (L-1)n - \sum_{i=1}^{L-1} J_i + 1\}$$

and if we let Y be of type B_m where $m = n - (L-1)n + \sum_{i=1}^{L-1} J_i$, then

$$\bigcup_{i=1}^{L-1} Ad(g_i)(\Omega_{X_i}) = \Omega_Y.$$

The eligibility condition ensures that

$$S_Y + S_{X_L} = 2 \left(n - (L-1)n + \sum_{i=1}^{L-1} J_i \right) + S_{X_L} \leq 2n,$$

so the pair (Y, X_L) is eligible. Complete the proof using the arguments of Prop. 3, but this time using Case 3 as X_L and Y are of opposite dominant types.

The argument is similar, but easier, if the Lie algebra is type $SU(n)$. We first check that (X'_1, \dots, X'_L) is eligible when (X_1, \dots, X_L) is eligible. This is clear if at most one X_i has $S_{X_i} = S_{X'_i}$. If two or more X_i have $S_{X_i} = S_{X'_i}$, then these two satisfy $S_{X_i} \leq n/2$ and because all $S_{X'} \leq n-2$, we have

$$\sum_{i=1}^L S_{X'_i} \leq 2(n/2) + (L-2)(n-2) \leq (L-1)(n-1),$$

proving eligibility.

Set $\Omega = \{FEE_1 - e_j : 2 \leq j \leq n\}$. We have $\Omega_{X_i} = \{FEE_1 - e_j : j > S_{X_i}\}$. Upon taking g_i suitable Weyl conjugates that permute letters we obtain

$$\bigcup_{i=1}^{L-1} Ad(g_i)\Omega_{X_i} = \Omega_Y.$$

where Y is an element of the torus of $SU(n)$ of type $SU(m)$ with $m = n - (L-1)n + \sum_{i=1}^{L-1} J_i$. The eligibility assumption ensures (X_L, Y) is an eligible pair and it is clearly not exceptional. Now complete the argument using the $L = 2$ case in the same manner as for type B_n and C_n . \square

The many exceptional pairs in D_n , ($n = 4$ in particular), cause complications in proving the theorem for type D_n . We will again prove the main theorem by an induction argument for $L = 2$, but it will be convenient to begin the argument with type D_5 . In the next lemma we will prove that all eligible, non-exceptional pairs in D_4 and D_5 are absolutely continuous. This will start the base case for us.

We will actually begin with D_3 . Usually D_n is defined for $n \geq 4$, but that is because D_3 is Lie isomorphic to type A_3 . As the problem of characterizing the L -tuples in type A_3 has already been done we can use this characterization, together with the induction step, Prop. 3, to handle most of the eligible, non-exceptional pairs in D_4 and D_5 . This approach will work whenever the reduced pair is known to be an absolutely continuous pair (in D_3 or D_4 , respectively). There will still be a few remaining pairs to consider and these will be handled directly by verifying Wright's criteria for absolute continuity, Thm. 2.

LEMMA 6. *All the eligible, non-exceptional pairs in D_4 and D_5 are absolutely continuous.*

PROOF. As explained above, we begin the proof by considering D_3 . Under the Lie isomorphism between D_3 and A_3 , any subsystem of type D_2 in D_3 is isomorphic to one of type $A_1 \times A_1$, type D_1 is isomorphic to one of type A_0 (or $SU(1)$) and types $SU(j)$ for $j = 1, 2, 3$ are unchanged under such an isomorphism. With this observation and the criteria for absolute continuity already known for the Lie algebra of type A_3 , it is easy to check that all pairs (X, Y) in D_3 are absolutely continuous except those of type (D_2, D_2) , $(D_2, SU(3))$, $(SU(3), SU(3))$ and $(SU(3), SU(2))$, the first two of these being not eligible and the latter two, exceptional.

Case D_4 : Prop. 3 guarantees that all eligible, non-exceptional pairs, (X, Y) , in D_4 are absolutely continuous, except when the reduced pair, (X', Y') , is one of the four pairs listed above. Furthermore, because we have already seen that the pair (X', Y') is eligible whenever (X, Y) is an eligible, non-exceptional pair, we will only need to give a special argument for those pairs (X, Y) where X' is type $SU(3)$ and Y' is either type $SU(3)$ or $SU(2)$ (the latter being type $SU(2) \times D_1$ or $SU(2) \times SU(1)$).

Thus we are left to study the pairs (X, Y) where X is of type $SU(4)$ and Y is one of type $SU(4)$, type $SU(3)$ (to be more precise, either type $SU(3) \times D_1$ or $SU(3) \times SU(1)$), type $SU(2) \times D_2$ or $SU(2) \times SU(2)$. However, these are all exceptional pairs except when X is type $SU(4)$, Y is of type $SU(2) \times SU(2)$ and Φ_Y is not Weyl conjugate to a subset of Φ_X .

To prove this last pair is absolutely continuous, we verify the criteria of Thm. 2 (with $X_1 = X$ and $X_2 = Y$) and follow the notation there. Here we have $|\Phi| = 24$ and $|\Phi_{X_1}| + |\Phi_{X_2}| = 12 + 4 = 16$. The rank 3, root subsystems, Ψ of D_4 , are those of type D_3 , $SU(4)$ (two non-Weyl conjugate subsystems) and $D_2 \times SU(2)$.

When Ψ is type $D_2 \times SU(2)$, then $|\Psi| = 6$. Thus we even have $|\Phi| - |\Psi| - 1 \geq |\Phi_{X_1}| + |\Phi_{X_2}|$, so (3.3) clearly holds. When Ψ is type D_3 , then $|\Psi| = 12$. However, $|\Phi_{X_1} \cap \sigma(\Psi)| = 6$ and $|\Phi_{X_2} \cap \sigma(\Psi)| \geq 2$ for all choices of $\sigma \in W$ because $\sigma(\Psi)$ must contain $\pm e_i \pm e_j, \pm e_i \pm e_k, \pm e_j \pm e_k$ for three choices of letters i, j, k . Thus the LHS of (3.3) is 11, while the RHS is at most 8.

Now assume Ψ is type $SU(4)$. First, suppose Ψ is Weyl conjugate to the set of annihilators of X . Since we need to calculate the intersection of Φ_{X_j} with all Weyl conjugates of Ψ , there is no loss of generality in assuming $\Phi_{X_1} = \Psi = \{e_i - e_j : 1 \leq$

$i \neq j \leq 4\}$. By assumption, Φ_{X_2} is not Weyl conjugate to a subset of Φ_{X_1} , thus there is also no loss of generality in assuming $\Phi_{X_2} = \{\pm(e_1 - e_2), \pm(e_3 + e_4)\}$.

The reader can check that $|\Phi_{X_1} \cap \sigma(\Psi)|$ is minimal when we take the choice of $\sigma \in W$ that switches two signs and in this case $|\Phi_{X_1} \cap \sigma(\Psi)| = 4$. Similarly, it can be shown that if σ is any Weyl conjugate, then $|\Phi_{X_2} \cap \sigma(\Psi)| \geq 2$, so that again the LHS of (3.3) is 11 and the RHS is at most 10.

Finally, suppose Ψ is not Weyl conjugate to Φ_{X_1} . Without loss of generality we can assume Ψ is as before and $\Phi_{X_1} = \{e_i - e_j, \pm(e_4 + e_j) : 1 \leq i \neq j \leq 3\}$. Again $|\Phi_{X_1} \cap \sigma(\Psi)|$ is minimal when σ is the Weyl element that switches two signs, but in this case $|\Phi_{X_1} \cap \sigma(\Psi)| = 6$. This is already enough to establish (3.3) and completes the argument that (X, Y) is an absolutely continuous pair.

Case D_5 : Again, Prop. 3 implies we only need to study the eligible non-exceptional pairs, (X, Y) in D_5 , where the reduced pair has X' of type $SU(4)$ and Y' one of type $SU(4)$, $SU(3)$, $SU(2) \times D_2$, or $SU(2) \times SU(2)$. Since the pairs $(SU(5), SU(5))$ and $(SU(5), SU(4))$ are exceptional and the pair $(SU(5), D_3 \times SU(2))$ is not eligible, this reduces the problem to the study of the pairs (X, Y) where X is of type $SU(5)$ and Y is either of type $SU(3) \times D_2$ or $SU(3) \times SU(2)$. Further, since the set of annihilators of an element of type $SU(3) \times SU(2)$ is contained in the set of annihilators of an element of type $SU(3) \times D_2$, it will suffice to prove that the pair $(SU(5), SU(3) \times D_2)$ is absolutely continuous.

For this, we again use Thm. 2. In D_5 , the rank 4 root subsystems Ψ which we must study are those of type D_4 , $SU(5)$, $D_3 \times SU(2)$ and $D_2 \times SU(3)$, with cardinalities 24, 20, 14 and 10, respectively. The cardinality of Φ is 40, while $|\Phi_{X_1}| = 20$ and $|\Phi_{X_2}| = 10$.

Let Λ be a root subsystem of type D_2, D_3 or D_4 in D_5 . It is easy to see that if Λ is a root subsystem of type D_j in D_5 , with $j = 2, 3, 4$, then $|\Phi_{X_1} \cap \Lambda| = \frac{1}{2}|\Lambda|$. Furthermore, $|\Phi_{X_2} \cap \Lambda| = 6$ whenever Λ is type D_4 . Since the action of a Weyl element preserves the type of a root subsystem, these calculations can be used to show that (3.3) holds if Ψ is type D_4 , $D_3 \times SU(2)$ or $D_2 \times SU(3)$.

When Ψ is type $SU(5)$, then $|\Phi_{X_1} \cap \sigma(\Psi)| \geq 8$ for all σ (with the minimum occurring when σ is two sign changes). Moreover, $|\Phi_{X_2} \cap \sigma(\Psi)| \geq 4$ so that again (3.3) is satisfied. This shows that the pair $(SU(5), SU(3) \times D_2)$ is absolutely continuous and completes the base case arguments. \square

Further complications arise with type D_n because of the fact that when X is of type $SU(n)$, then μ_X^2 is not absolutely continuous. We have already seen this complication in the proof of Prop. 3 (when $L = 2$), but it presents further difficulties when $L > 2$. To handle this, we introduce the following terminology for the remainder of the proof.

DEFINITION 5. We will say that X is **almost dominant SU type** if X is type $D_J \times SU(s_1) \times \cdots \times SU(s_t)$ where $J \leq \sum s_i$.

Of course, if X is dominant SU type, then it is almost dominant SU type. However, X is also almost dominant SU type if X is dominant D type, but X' is dominant SU type, for instance. If X is almost dominant SU type and not type $SU(n)$, then [9, Thm. 8.2], implies $\mu_X^2 \in L^2$. Here are some additional properties.

LEMMA 7. Suppose X_1, X_2 are almost dominant SU type in D_n and $X_3 \neq 0$.

- (i) If neither X_1 nor X_2 are type $SU(n)$, then $\mu_{X_1} * \mu_{X_2} \in L^2$.
- (ii) If X_1 and X_2 are both type $SU(n)$ and X_3 is not, then $\mu_{X_1} * \mu_{X_2} * \mu_{X_3} \in L^2$.

(iii) More generally, if X_1 is type $SU(n)$ and X_2 is not, then $\mu_{X_1} * \mu_{X_2} * \mu_{X_3} \in L^2$.

(iv) If $n \geq 5$ (or $n = 4$) and X_3 (and X_4) is almost dominant $SU(n)$ type, then $\mu_{X_1} * \mu_{X_2} * \mu_{X_3}(*\mu_{X_4}) \in L^2$.

PROOF. (i) follows from [9] as remarked above. The fact that it is absolutely continuous, which is actually all we will need for our application, also follows from the $L = 2$ part of the proof of the main theorem since (X_1, X_2) is an eligible, non-exceptional pair.

(iv) holds similarly from [9] since $\mu_X^3 \in L^2$ whenever X is almost dominate SU type and $n \geq 5$, and $\mu_X^4 \in L^2$ when $n = 4$. (Alternatively, absolute continuity can be checked from Theorem 2.)

For (ii) and (iii) we proceed by induction on n , noting that according to the main theorem, as already established for all $L \geq 2$ in the Lie algebras of type A_n , all triples in A_3 (equivalently, D_3) are absolutely continuous, except when all three are type $SU(3)$.

Now assume $n \geq 4$. We put

$$\Omega = \{FEe_1 \pm e_j : j = 2, \dots, n, F = R, I\}.$$

(ii): Here X'_1, X'_2 will be of type $SU(n-1)$ in D_{n-1} , while X'_3 is not, so the induction hypothesis applies. Without loss of generality we can assume

$$\Omega_{X_1} = \{FEe_1 + e_j : j \geq 2, F = R, I\}$$

and Ω_{X_2} either coincides with Ω_{X_1} or

$$\Omega_{X_2} = \{FEe_1 + e_j, FEe_1 - e_n : j \leq n-1, F = R, I\}.$$

As X_3 is not of type $SU(n)$, Ω_{X_3} contains $\{FEe_1 \pm e_n\}$. Let g be the Weyl conjugate changing the signs of $2, \dots, n-1$ (and n if needed to be an even sign change). Then $\Omega_{X_1} \cup Ad(g)(\Omega_{X_2})$ contains all of Ω except for possibly $\{FEe_1 - e_n : F = R, I\}$. If $\Omega_0 = \{FEe_1 + e_n\}$, we have

$$\Omega_{X_1} \cup Ad(g)(\Omega_{X_2}) \cup (\Omega_{X_3} \setminus \Omega_0) = \Omega.$$

Taking $M = REe_1 + e_n$ one can verify that the hypotheses of the general strategy, Prop. 2, are all satisfied. Consequently, (X_1, X_2, X_3) is absolutely continuous.

(iii): We define X'_1 and X'_3 as usual, but will re-define X'_2 so that it continues to be of almost dominant SU type and not type $SU(n-1)$ (so that we will be able to apply the induction hypothesis). This can be achieved by defining X'_2 to be type $D_{J-1} \times SU(s_1) \times \dots \times SU(s_t)$ if X_2 is type $D_J \times SU(s_1) \times \dots \times SU(s_t)$ with $J > 1$, or defining X'_2 to be type $D_J \times SU(s_{1-1}) \times \dots \times SU(s_t)$ if $J = 0$ or 1 and $s_1 = \max s_j$. The fact that X_2 is almost dominant SU type ensures that $J \leq n/2$, so whether X_2 is dominant SU type or not, $S_{X_2} \leq n$ and thus (X_1, X_2) is an eligible pair. Further, X_1, X_2 are not both of type $SU(n)$.

The arguments given in Prop. 3, Case 2 or 3 depending on the situation, can be applied to prove there is some $g \in D_{n-1}$ such that $sp\{\Omega_{X_1} \setminus \Omega_0, Ad(g)(\Omega_{X_2})\} = sp\Omega$ where Ω_0 is taken to be the choice of $FEe_1 + e_n$ or $FEe_1 - e_n$ that belongs to Ω_{X_1} . Therefore

$$sp\{\Omega_{X_1} \setminus \Omega_0, Ad(g)(\Omega_{X_2}), \Omega_{X_3}\} = sp\Omega.$$

Now take $M = REe_1 \pm e_n$ (depending on the choice of Ω_0) and apply the general strategy. \square

We are now ready to conclude the proof of Theorem 1 by completing the proof of sufficiency for absolute continuity in type D_n .

PROOF OF THEOREM 1 CONTINUED. Sufficient conditions for Absolute continuity for Lie type D_n :

Case $L = 2$. Lemma 6 starts the induction argument for type D_n . Now, inductively assume that all eligible, non-exceptional pairs in D_{n-1} , with $n \geq 6$, are absolutely continuous. By Lemma 3, the pair (X', Y') is eligible. If it is an exceptional pair, then it must be either of type $(SU(n-1), SU(n-1))$ or type $(SU(n-1), SU(n-2))$ (where the $SU(n-2)$ could be type $SU(n-2) \times D_1$ or $SU(n-2) \times SU(1)$). But then (X, Y) must also have been an exceptional pair in D_n , which is a contradiction. By the induction assumption, (X', Y') is an absolutely continuous pair and hence Prop. 3 implies that (X, Y) is absolutely continuous.

Case $L \geq 3$. Again, we give an induction argument. The base case, D_4 , will be discussed at the conclusion of the proof. So assume $n \geq 5$ and (X_1, \dots, X_L) is an eligible L -tuple in D_n . We note that there are no exceptional L -tuples in D_n for $n \geq 5$ when $L \geq 3$.

We will take

$$\Omega = \{FEe_1 \pm e_j : j = 2, \dots, n\}.$$

More care is needed in this situation than for the Lie algebras of type B_n and C_n , since the fact that $\mu_X^2 \notin L^2$ when X is of type $SU(n)$ means, for example, that we cannot immediately assume that at most one X_i is dominant SU type, as we did in the argument for those Lie types. Here is where Lemma 7 will be useful.

If three or more X_i are dominant SU type, then the induction argument is not even necessary as Lemma 7 (iv) implies that their convolution is already in L^2 and hence is absolutely continuous.

If two X_i (say, X_1, X_2) are both dominant SU type and some X_j , say X_3 , is not, then we call upon one of the first three parts of the lemma.

So we can assume there is at most one X_i that is dominant SU type, say X_1 . If there is some X_j , other than X_1 , with X'_j of dominant SU type, then X_j is almost dominant SU and not type $SU(n)$. Apply the appropriate part of Lemma 7 with X_1, X_2 equal to this X_j , and X_3 any other X_i .

If all X'_j , other than $j = 1$, remain dominant D type, then the calculations used in the type B_n or C_n case show that (X'_1, \dots, X'_L) is an eligible, non-exceptional tuple in D_{n-1} . For the induction step we argue in the same fashion as we did for the Lie algebras of type B_n or C_n in the same situation.

Finally, assume all X_j are dominant D type. If two or more X'_j are dominant SU type, then the corresponding two X_j are almost dominant SU type and not of type $SU(n)$. Their convolution is even in L^2 . If at most one X'_j is dominant SU type, then (X'_1, \dots, X'_L) is an eligible tuple, so the induction hypothesis applies. The induction step is the same as for the corresponding situation with type B_n or C_n .

To conclude, we must establish the base case, $n = 4$. Since $\mu_X^4 \in L^2$ for any non-trivial X in the Lie algebra of type D_4 , every L -tuple with $L \geq 4$ is an absolutely continuous tuple.

So we can assume $L = 3$. The induction argument above can be applied to (X_1, X_2, X_3) provided at most one X_j is dominant SU type, hence for such triples it suffices to check that (X'_1, X'_2, X'_3) in D_3 is an absolutely continuous triple. But

this follows from the main theorem for type A_n , since all triples in A_3 , except when all X_i are type $SU(3)$, are absolutely continuous.

If two or three X_j are dominant SU type, but at least one X_i is not of type $SU(4)$, Lemma 7 gives the result.

If all three X_i are type $SU(4)$ and their annihilating root systems are Weyl conjugate, then the triple, (X_1, X_2, X_3) , is exceptional. Thus we can assume the annihilating root systems are not Weyl conjugate. As the arguments are symmetric, there is no loss of generality in assuming that the set of annihilating roots for X_1 coincides with that of X_2 and is given by

$$\Phi_{X_1} = \Phi_{X_2} = \{e_i - e_j : 1 \leq i \neq j \leq 4\},$$

while

$$\Phi_{X_3} = \{e_i - e_j, \pm(e_4 + e_k) : 1 \leq i \neq j \leq 3, k = 1, 2, 3\}.$$

We will again call upon Thm. 2 to check the absolute continuity of the triple. The root systems Ψ of rank 3 that we must consider are those of type D_3 , $SU(4)$ (two non-Weyl conjugate root subsystems) and $D_2 \times SU(2)$.

Of course, $\sum_{i=1}^3 |\Phi_{X_i}| = 36$ and $|\Phi| = 24$ when Φ is the root system of D_4 . When Ψ is type $D_2 \times SU(2)$, then $|\Psi| = 6$, and as Ψ intersects non-trivially any root subsystem of type $SU(4)$, the inequality (3.3) is clear. When Ψ is type D_3 it is easy to see that $|\Phi_{X_i} \cap \sigma(\Psi)| \geq 6$ for each i and any Weyl element σ .

When Ψ is type $SU(4)$, then $|\Phi_{X_i} \cap \sigma(\Psi)| \geq 4$. However, as noted in the $L = 2$ argument, if Ψ and Φ_{X_i} are non-Weyl conjugate subsystems of type $SU(4)$, then this lower bound can be improved to $\hat{6}$. Consequently, $2(|\Phi| - |\Psi|) - 1 = 23$, while the right hand side of (3.3) is at most $36 - (4 + 4 + 6) = 22$, so the inequality holds.

This completes the base case argument and hence the proof of Theorem 1. \square

7. Applications

7.1. Consequences of the Main Theorem. An element $X \in \mathfrak{t}_n$ is said to be *regular* if its set of annihilating roots is empty. These would be the elements of type $SU(1) \times \cdots \times SU(1)$ (in any Lie algebra) or $D_1 \times SU(1) \times \cdots \times SU(1)$ in type D_n , and hence have $S_X = 1$ or 2 . In [7] it was shown that the convolution of the orbital measures of any two regular elements is absolutely continuous. The methods used there could be used to prove, more generally, that the convolution of any orbital measure with the orbital measure of a regular element is absolutely continuous. Our theorem shows that more is true.

COROLLARY 2. *Let X, Y be non-zero elements in the Lie algebra of type B_n , C_n , or D_n . If $S_Y \leq 2$, then $\mu_X * \mu_Y$ is absolutely continuous and $O_X + O_Y$ has non-empty interior, except if (X, Y) is the exceptional pair $(SU(4), SU(2) \times SU(2))$ in D_4 where the annihilating roots of Y are a subset of a Weyl conjugate of those of X .*

PROOF. This is immediate from the theorem since any non-zero X has $S_X \leq 2(n-1)$ (with equality only if X is type B_{n-1} (C_{n-1} or D_{n-1})). \square

COROLLARY 3. *If (X_1, \dots, X_L) is an eligible, non-exceptional L -tuple of matrices in any of the classical Lie algebras, then there are unitarily similar matrices, $g_i^{-1} X_i g_i$, with the property that $\sum_{i=1}^L g_i^{-1} X_i g_i$ has distinct eigenvalues.*

PROOF. This follows from the main theorem because any subset of these matrix groups with non-empty interior must contain an element with distinct eigenvalues. Indeed, the elements with distinct eigenvalues are dense. \square

On the other hand, if X_i are matrices in one of the classical Lie algebras and there are unitarily similar matrices, $g_i^{-1}X_i g_i \in O_{X_i}$, with the property that $\sum_{i=1}^L g_i^{-1}X_i g_i$ has distinct eigenvalues, then $\sum O_{X_i}$ contains an element Y with $S_Y \leq 2$. (Indeed, Y is either type $SU(1)$ or type B_1, C_1 or D_1 .) As noted in the first corollary, $O_X + O_Y$ has non-empty interior for any $X \neq 0$ and thus $\mu_{X_1} * \dots * \mu_{X_L} * \mu_X$ is absolutely continuous for any $X \neq 0$. It would be interesting to characterize the L -tuples for which $\sum O_{X_i}$ contains a matrix with distinct eigenvalues.

It is known that any n -fold sum of non-trivial orbits in B_n, C_n or D_n has non-empty interior. More can be said.

COROLLARY 4. *Let $n \geq 5$. If X_i are non-zero elements in B_n (C_n or D_n) for $i = 1, \dots, n-1$, then $O_{X_1} + \dots + O_{X_{n-1}}$ has empty interior if and only if all X_i are type B_{n-1} (C_{n-1} or D_{n-1}).*

PROOF. Suppose some X_i , say X_{n-1} , is not type B_{n-1} . Then $S_{X_{n-1}} \leq 2(n-2)$. As all $S_{X_i} \leq 2(n-1)$,

$$\sum_{i=1}^{n-1} S_{X_i} \leq 2(n-1)(n-2) + 2(n-2) \leq 2n(n-2).$$

Thus (X_1, \dots, X_{n-1}) is eligible and non-exceptional and hence the sum of the orbits has non-empty interior. Since all root subsystems of type B_{n-1} are Weyl conjugate, the converse follows from the fact that if X is type B_{n-1} , then μ_X^{n-1} is not absolutely continuous [9]. \square

We leave it as an exercise for the reader to determine the choice of $n-1$ tuples that are not absolutely continuous when $n \leq 4$ and in type A_n .

We note that for type A_n, B_n and C_n our proof required the use of [9] only to start the induction process. In the proof given in [9] an induction argument was also used and the base cases were simply done directly. That approach could have been taken here, as well. For type D_n our proof also used [9] to establish that when X was type $SU(n)$, then μ_X^3 was absolutely continuous for $n \geq 5$ and μ_X^4 was absolutely continuous when $n = 4$. In fact, the argument that was given there for these special types actually showed that Theorem 2 was satisfied. Thus, our theorem gives another way to deduce the formulas of [9]. For example, we have

COROLLARY 5. *Suppose \mathfrak{g} is type B_n and X is type $B_J \times SU(s_1) \times \dots \times SU(s_m)$.*

(i) *If X is dominant B type, then μ_X^L is absolutely continuous (and $(L)O_X$ has non-empty interior) if and only if $L \geq n/(n-J)$.*

(ii) *If X is dominant SU type, then μ_X^2 is absolutely continuous.*

Similar statements can be made for the other types, taking into account the exceptional cases.

REMARK 5. (i) We have not been able to determine if the pair of type $(SU(n), SU(n-1))$ in D_n for $n \geq 6$ is absolutely continuous. Computer results suggest that it is not for at least $n = 6, 7$. We remark that Prop. 3 shows that if such a pair is absolutely continuous for any n , say $n = n_0$ then, being an eligible pair, it is absolutely continuous for all $n > n_0$.

(ii) It remains open to solve the analogous problem in the exceptional Lie algebras, those of type G_2 , F_4 , E_6 , E_7 , or E_8 . In [10] the minimal $k(X)$ so that $\mu_X^{k(X)}$ is absolutely continuous was determined for each X in the compact exceptional Lie algebras. Although the abstract root theory machinery can be applied in this setting, there is no underlying classical matrix algebra from which to derive the necessary conditions.

7.2. Orbital measures on Conjugacy classes in Compact Lie Groups.

A related, but more challenging problem, is to determine which L -tuples, $(x_1, \dots, x_L) \in G^L$, have the property that $\mu_{x_1} * \dots * \mu_{x_L}$ is absolutely continuous with respect to Haar measure on the group G , when μ_x is the probability measure, invariant under the conjugation action of G on itself, and supported on the conjugacy class generated by x , $C_x = \{g^{-1}xg : g \in G\}$. In [8, Thm. 9.1], the minimum integer $k(x)$ for which $\mu_x^{k(x)}$ is absolutely continuous was determined for all the classical Lie groups. The number $k(x)$ depended on the type of the set of annihilating roots of x , where in this setting by the set of annihilating roots we mean

$$\Phi_x := \{\alpha \in \Phi : \alpha(x) \equiv 0 \pmod{2\pi}\}.$$

Again, by the type of x , we will mean the type of Φ_x .

This was extended by Wright [23] to convolution products of (possibly) different μ_x in the case of $SU(n)$, obtaining the same characterization as for the Lie algebra problem. In this subsection, we will obtain a similar result for all the classical Lie groups whenever the group elements $x_i = \exp X_i$ where $X_i \in \mathfrak{g}$ and $x_i \in G$ have the same type.

We need the following preliminary result, analogous to Prop. 1. Given $x \in G$, we let

$$\mathcal{N}_x := \{RE_\alpha, IE_\alpha : \alpha(x) \neq 0 \pmod{2\pi}\}.$$

LEMMA 8. (c.f. [18], [23]) *The measure $\mu_{x_1} * \dots * \mu_{x_L}$ on G_n is absolutely continuous with respect to Haar measure on G_n if and only if any of the following hold:*

- (i) *The set $\prod_{i=1}^L C_{x_i} \subseteq G_n$ has non-empty interior;*
- (ii) *The set $\prod_{i=1}^L C_{x_i} \subseteq G_n$ has positive measure;*
- (iii) *There exists $g_i \in G_n$ with $g_1 = Id$, such that*

$$sp\{Ad(g_i)\mathcal{N}_{x_i} : i = 1, \dots, L\} = \mathfrak{g}_n.$$

PROPOSITION 4. *Let $x_1, \dots, x_L \in G_n$ and assume $x_i = \exp X_i$ for some $X_i \in \mathfrak{g}_n$ where x_i and X_i have the same type. Then $\mu_{x_1} * \dots * \mu_{x_L}$ is absolutely continuous with respect to Haar measure on G_n if and only if $\mu_{X_1} * \dots * \mu_{X_L}$ is absolutely continuous with respect to Lebesgue measure on \mathfrak{g}_n . Moreover, $\prod_{i=1}^L C_{x_i}$ has non-empty interior in G_n if and only if $\sum_{i=1}^L O_{X_i}$ has non-empty interior in \mathfrak{g}_n .*

PROOF. If x_i and X_i are of the same type, then $\mathcal{N}_{x_i} = \mathcal{N}_{X_i}$. Consequently,

$$sp\{Ad(g_i)\mathcal{N}_{x_i} : i = 1, \dots, L\} = sp\{Ad(g_i)\mathcal{N}_{X_i} : i = 1, \dots, L\}$$

and thus $\mu_{x_1} * \cdots * \mu_{x_L}$ is absolutely continuous if and only if $\mu_{X_1} * \cdots * \mu_{X_L}$ is absolutely continuous. The latter statement holds as absolute continuity is equivalent to non-empty interior of either the product of conjugacy classes or the sum of orbits, depending on the setting. \square

REMARK 6. If $x_i = \exp X_i$ and $\mu_{X_1} * \cdots * \mu_{X_L}$ is not absolutely continuous, then it still follows $\mu_{x_1} * \cdots * \mu_{x_L}$ is not absolutely continuous. We simply note that always $\Phi_{X_i} \subseteq \Phi_{x_i}$.

Consider the Lie group $SU(n)$. Every conjugacy class contains a diagonal matrix so in studying the measure μ_x there is no loss in assuming

$$x = \text{diag}(\exp ia_1, \dots, \exp ia_n)$$

where $a_j \in [0, 2\pi)$ and $\sum a_j \equiv 0 \pmod{2\pi}$. Notice that $x = \exp X$ where $X = \text{diag}(ia_1, \dots, ia_n)$ belongs to $su(n)$. The root $\alpha = e_j - e_k$ acts on x (and X) by $\alpha(x) = a_j - a_k$. Thus $\Phi_x = \Phi_X$ and so the Proposition applies to all L -tuples in $SU(n)$.

This is not true for the other classical Lie groups. For example, in $SO(2n+1)$ (type B_n) there is an element x with $\Phi_x = \{e_i \pm e_j : 1 \leq i \neq j \leq n\}$, i.e., of type D_n . This type does not arise in the Lie algebra. Indeed, the only element $X \in so(2n+1)$ with $\Phi_X \supseteq \Phi_x$ is $X = 0$. The element x has the property that $\mu_x^{2n} \in L^1(G)$, but μ_x^{2n-1} is singular with respect to Haar measure on G . In contrast, any X with $\exp X = x$ has $\mu_X^2 \in L^2(\mathfrak{g})$.

These additional (and often more complicated) types of elements that can arise in the Lie groups make the problem of characterizing absolute continuity of orbital measures on Lie groups more challenging than for Lie algebras.

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